Corner Polyhedra and Maximal Lattice-free Convex Sets : A Geometric Approach to Cutting Planes

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Abstract

Corner Polyhedra were introduced by Gomory in the early 60s and were studied by Gomory and Johnson. The importance of the corner polyhedron is underscored by the fact that almost all "generic" cutting planes, both in the theoretical literature as well as ones used in practice, are valid for the corner polyhedron. So the corner polyhedron can be viewed as a unifying structure from which many of the known cutting planes can be derived. Moreover, the corner polyhedron has recently attracted a lot of attention as a potential source of new kinds of cutting planes.

It has also been observed that valid inequalities for the corner polyhedron have an intimate connection with maximal lattice-free convex sets in $\mathbb{R}^m$. This connection provides a geometric way of understanding and analyzing these valid inequalities. Such an approach often yields new insights into properties of existing cutting planes, as well as provides a novel way of analyzing new families of cutting planes.

This thesis explores this interplay of maximal lattice-free convex bodies and cutting plane theory. This leads to the development of the theory of the so-called "multiple-constraint" cutting planes. These families of cutting planes have received significant attention in recent years with the ultimate aim of further improving state-of-the-art integer programming algorithms. The results presented in this thesis contribute towards the mathematical foundations of this new line of research and shed light on the potential power of these new cutting planes.

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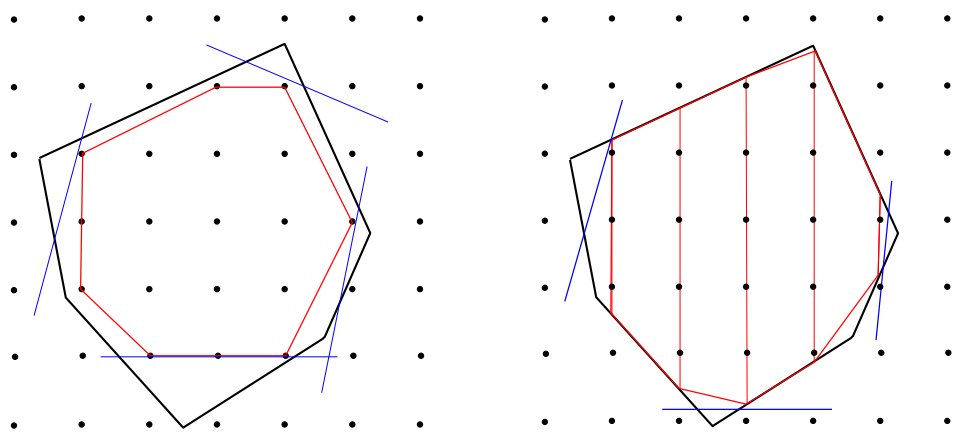
Chapter 1

Introduction

Mixed Integer Linear Programs (MILPs) are a general framework for modeling an extremely large number of problems in technological and scientific applications. Mathematically, they can be viewed as a generalization of the problem of solving systems of linear Diophantine equations, where we also allow inequalities in the system. Since in general MILPs fall in the NP-complete class of problems, we do not hope to obtain polynomial algorithms for solving them in general. Even so, algorithms have been developed which have other theoretical guarantees and also have been engineered to work remarkably well in practice. A very successful approach in this direction has been the *cutting-plane method*. Originally proposed by Gomory [42], this algorithm can be proved to terminate in finite time for pure integer problems (where all variables are required to be integer-valued). Moreover, the finite termination of this method can also be proved for mixed 0-1 problems as a consequence of Balas's sequential convexification result [12]. See also [11, 14]. Moreover, on the practical side, this method has seen a great revival in the 1990s with its effectiveness being truly exploited by clever engineering, robust implementation and increased computing power [15, 32]. Since then all successful commercial codes for solving MILPs (for instance, Xpress and Cplex) have relied heavily on the *branch-and-cut* framework and cutting planes are an extremely effective method for solving large MILPs today. In fact, MILPs which would have required years of computation time 20 years ago can now be solved in seconds today (see Bixby et al.[23] for a survey of these developments). In spite of this, there are several examples of MILPs (in particular combinatorial auctions and sports scheduling) which are beyond the scope of current state-of-the-art software. Advances in cutting plane and integer programming theory would be a main factor in achieving

such speed-ups. This is one of the major motivations behind recent investigations in this field.

Moreover, cutting plane theory makes for very elegant mathematics. Consider a system of inequalities $Ax \leq b$ where x is a vector of variables $x_1, \ldots, x_n$, A is an $m \times n$ matrix and b is a $m \times 1$ column vector. In other words, it is a system of m inequalities in n variables. We wish to find integral solutions to the system, i.e. $x \in \mathbb{Z}^n$ such that $Ax \leq b$. The term *integer hull* is used to refer to the convex hull of all such x. The basic idea behind cutting plane theory is to approximate the integer hull as well as possible by introducing additional linear inequalities into the system. These linear inequalities are such that every solution of the original system satisfy the new inequalities also. Such linear inequalities are classically called *valid inequalities* or *cutting planes* (a.k.a cuts) for the solution set. We will define these notions formally in Chapter 2.



(a) Both variables are integer valued (b) One variable is continuous and one is integer valued

Figure 1.1: Illustration for Cutting Planes

The set $\{x \in \mathbb{R}^n : Ax \leq b\}$, where we have relaxed the integrality constraints on the variables, will be referred to as the *linear relaxation* of the original problem. Such a set is called a *polyhedron* in $\mathbb{R}^n$. The interesting cutting planes are ones which cut off some portions of the linear relaxation. Figure 1.1(a) illustrates this geometric idea in two dimensions. The black lines define the original inequalities of the system and hence form the linear relaxation. The blue lines are examples of cutting planes or valid inequalities. The red lines show the integer hull, i.e. the convex hull of the integer solutions. Mixed Integer Linear Programs (MILPs) add the further generality that only a subset of the variables $x_1, \ldots, x_n$ are required to be integer-valued. The notions of integer hull, linear relaxation and cutting planes generalize in a straightforward manner to such problems. See Figure 1.1(b) - the light red lines show the entire solution set, the dark red lines show the integer hull.

1.1 Motivation

Until the mid-90s, algorithms for solving MILPs did not fare well in practice. MILP models for real-world problems were usually intractable by existing software and these methods were good only for small, artificial instances. A major factor in the enormous improvement in solving MILPs in the past 15-20 years was the proper engineering of a class of cutting planes, known as the *split cuts*. Examples in this class are Gomory mixed-integer cuts, Reduce-and-split cuts, and MIR cuts. The key property that these cuts share is that they can essentially be derived by analyzing a single constraint of the MILP system. It is possible to generalize this procedure to derive cutting planes from two or more constraints. This generalization hopes to obtain stronger cutting planes (the notion of stronger will be discussed later in this thesis), since we are using more information from the MILP system to derive the inequalities.

The grand motivation behind this program of research is to achieve a second breakthrough in the efficiency of MILP solvers, guided by the intuition that these new kinds of cuts are quite different from the "single-constraint" cuts. This could be the next leap in solving general real-world problems which fall under the MILP framework.

On the more theoretical side, the results of this thesis show that techniques from convex analysis and lattice theory are very useful for analyzing these "multiple-constraint" cutting planes. We hope that this will encourage even more interaction between these fields and lead to the discovery of new and beautiful mathematical

structures.

1.2 Contribution and Results

We summarize the contributions of this thesis in the following points.

- **Maximal Lattice-free Convex sets and Cutting Planes**

  Results are obtained which form an integral part of the mathematical foundations for the theory of multiple-constraint cutting planes. Much of this work generalizes and extends classical results of Lovasz [52] and Balas [11, 13] and draws deep connections between cutting planes and lattice theory. Specifically, we develop the geometric theory of *maximal lattice-free convex sets* to study cutting planes. We characterize precisely the structure of maximal lattice-free convex sets in linear subspaces of $\mathbb{R}^n$. Using this theory, we analyze linear systems proposed by Borozan and Cornuejols [25] and Dey and Wolsey [38] for the study of cutting planes. This theory provides a novel geometric lens to analyze cutting planes and is extremely useful for investigating many of their properties. The mathematical techniques involved in this work span classical convex analysis and polyhedral theory.

- **Relative Strength of Two-Constraint Cutting Planes**

  We provide theoretical guarantees for the superiority of two-constraint cuts over the single-constraint cuts. We study a system defined by two equations with two free integer variables and any number of nonnegative continuous variables, first proposed by Andersen et al. [8]. We show that, in a well-defined mathematical sense, two-constraint cuts provide a good approximation of the integer hull. On the other hand, the approximation produced by single-constraint cuts may be arbitrarily bad. To the best of our knowledge, these results are the first of their kind regarding general cutting plane families for MILPs. Most of the arguments in these proofs are purely geometric in nature and demonstrate the power of using maximal lattice-free sets for the analysis of cutting planes.

- **Infinite Group Problem**

  Another result presented in this thesis contributes to our understanding of the so-called *infinite group problem*. Gomory and Johnson introduced this

problem for the purpose of studying cutting planes derived from a single constraint. We settle a conjecture made by Gomory and Johnson in [46] about the existence of facets for the infinite group problem which are not piecewise linear.

- **Computational Study**

  We also present results from a computational study we made on the usefulness of two-constraint cuts. This project not only sheds some light on the relative power of two-constraint cuts, but also leads to a deeper understanding of the issues behind the engineering of cutting planes. We hope these observations will benefit researchers interested in building cutting plane software.

1.2.1 Outline of the Thesis

We present the above results in the subsequent chapters of this thesis. In Chapter 2, we collect classical results from the theory of convex polyhedra, lattice theory, cutting planes, convex and functional analysis needed for the exposition of the results. We also give a short introduction to the concept of the Corner Polyhedron in this chapter. In Chapters 3 and 4, we develop the theory of maximal lattice-free sets and present the results which connect these structures to cutting planes for Integer Programs. We then present our theoretical results on the relative strength of two-constraint cutting planes as compared to single-constraint cutting planes in Chapter 5. In the next chapter, we present our results from the computational study of these two-constraint cuts, motivated by the theoretical results. Chapter 7 addresses the facet problem for the infinite group problem. Here we settle a conjecture (by exhibiting a counterexample) made by Gomory and Johnson in [46].

Chapter 2

Lattices, Convexity, Corner Polyhedra and Intersection Cuts

This chapter reviews classical work from lattice theory, convex analysis, polyhedral theory and cutting planes, needed for the exposition of the results in the subsequent chapters. Most of the theorems are stated in this chapter without proof.

Given $X \subset \mathbb{R}^n$, we denote by $\langle X \rangle$ the linear space generated by the vectors in X. The underlying field is $\mathbb{R}$ in this thesis. Given a linear subspace L of $\mathbb{R}^n$, we denote by $L^\perp$ the orthogonal complement of L. Given a set $S \subseteq \mathbb{R}^n$, the *orthogonal projection of S onto $L^\perp$* is the set

$$\mathrm{proj}_{L^\perp}(S) = \{v \in L^\perp \mid v + w \in S \text{ for some } w \in L\}.$$

Before we begin, we state Zorn's Lemma which will be used later.

Lemma 1 (Zorn's Lemma)**.** *Consider any partially ordered set P. If every chain in P has an upper bound, then there exists a maximal element in P.*

2.1 Lattices

Definition 2. *An additive group Λ of $\mathbb{R}^n$ is said to be* finitely generated *if there exist vectors $a_1, \ldots, a_m \in \mathbb{R}^n$ such that $\Lambda = \{\lambda_1 a_1 + \ldots + \lambda_m a_m \mid \lambda_1, \ldots, \lambda_m \in \mathbb{Z}\}$.*

If a finitely generated additive group Λ of $\mathbb{R}^n$ can be generated by linearly independent *vectors $a_1, \ldots, a_m$, then Λ is called a* lattice of the linear space $\langle a_1, \ldots, a_m \rangle$. *The set of vectors $a_1, \ldots, a_m$ is called a* basis *of the lattice Λ.*

The following standard result in lattice theory provides a useful equivalent definition of lattice (see Barvinok [17], p. 284 Theorem 1.4).

Theorem 3. *Let Λ be the additive group generated by vectors $a_1, \ldots, a_m \in \mathbb{R}^n$. Then Λ is a lattice of the linear space $\langle a_1, \ldots, a_m \rangle$ if and only if there exists $\varepsilon > 0$ such that $\|y\| \geq \varepsilon$ for every $y \in \Lambda \setminus \{0\}$.*

Definition 4. *Let Λ be a lattice of a linear space V of $\mathbb{R}^n$. Given a linear subspace L of V, we say that L is a* lattice-subspace of V *if there exists a basis of L contained in Λ.*

We will use the following result (see Barvinok [17], p. 284 problem 3).

Lemma 5. *Given a lattice-subspace L of V, the orthogonal projection of Λ onto $L^\perp$ is a lattice of $L^\perp \cap V$.*

The proof of this theorem relies on the fact that if L is a lattice-subspace of V and Λ is a lattice of V, then there exists an $\epsilon > 0$ such that every lattice point in $\Lambda \setminus L$ is at a distance at least ϵ from L (see Barvinok [17], p. 282, Lemma 1.3). A converse of this theorem is true, which we prove below. It relies on the following famous approximation theorem of Dirichlet.

Theorem 6 (Dirichlet)**.** *Given real numbers $\alpha_1, \ldots, \alpha_n, \varepsilon$ with $0 < \varepsilon < 1$, there exist integers $p_1, \ldots, p_n$ and q such that*

$$\left| \alpha_i - \frac{p_i}{q} \right| < \frac{\varepsilon}{q}, \text{ for } i = 1, \ldots, n, \text{ and } 1 \leq q \leq \varepsilon^{-1}. \tag{2.1}$$

The following is a consequence of Dirichlet's theorem.

Lemma 7. *Given $y \in \Lambda$ and $r \in V$, then for every $\varepsilon > 0$ and $\bar{\lambda} \geq 0$, there exists a point of Λ at distance less than ε from the half line $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$.*

Proof. First we show that, if the statement holds for $\bar{\lambda} = 0$, then it holds for arbitrary $\bar{\lambda}$. Given $\varepsilon > 0$, let Z be the set of points of Λ at distance less than ε from $\{y + \lambda r \mid \lambda \geq 0\}$. Suppose, by contradiction, that no point in Z has distance less than ε from $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$. Then Z is contained in $B_\varepsilon(0) + \{y + \lambda r \mid 0 \leq \lambda \leq \bar{\lambda}\}$. By Theorem 3, Z is finite, thus there exists an $\bar{\varepsilon} > 0$ such that every point in Z has distance greater than $\bar{\varepsilon}$ from $\{y + \lambda r \mid \lambda \geq 0\}$, a contradiction. So we only need to show that, given $\bar{\varepsilon}$, there exists at least one point of Λ at distance at most $\bar{\varepsilon}$ from $\{y + \lambda r \mid \lambda \geq 0\}$.

Let $m = \dim(V)$ and $a_1, \ldots, a_m$ be a basis of Λ. Then there exists $\alpha \in \mathbb{R}^m$ such that $r = \alpha_1 a_1 + \ldots, \alpha_m a_m$. Denote by A the matrix with columns $a_1, \ldots, a_m$, and define $\|A\| = \sup_{x\,:\,\|x\|\le 1} \|Ax\|$ where, for a vector v, $\|v\|$ denotes the Euclidean norm of v. Choose $\varepsilon > 0$ such that $\varepsilon < 1$ and $\varepsilon \le \bar{\varepsilon}/(\|A\|\sqrt{m})$. By Dirichlet's theorem, there exist $p \in \mathbb{Z}^m$ and $\lambda > 0$ such that

$$\|\alpha - \frac{p}{\lambda}\| = \sqrt{\sum_{i=1}^{m} \left|\alpha_i - \frac{p_i}{\lambda}\right|^2} \le \frac{\varepsilon\sqrt{m}}{\lambda} \le \frac{\bar{\varepsilon}}{\|A\|\lambda}.$$

Let $z = Ap + y$. Since $p \in \mathbb{Z}^m$, then $z \in \Lambda$. Furthermore

$$\|(y + \lambda r) - z\| = \|\lambda r - Ap\| = \|A(\lambda\alpha - p)\| \le \|A\|\|\lambda\alpha - p\| \le \bar{\varepsilon}.$$

□

We use the above result to prove the converse of Lemma 5.

Lemma 8. *If a linear subspace L of V is not a lattice-subspace of V, then for every $\varepsilon > 0$ there exists $y \in \Lambda \setminus L$ at distance less than ε from L.*

Proof. The proof is by induction on $k = \dim(L)$. Assume L is a linear subspace of V that is not a lattice-subspace, and let $\varepsilon > 0$. If $k = 1$, then, since the origin 0 is contained in Λ, by Lemma 7 there exists $y \in \Lambda$ at distance less than ε from L. If $y \in L$, then $L = \langle y \rangle$, thus L is a lattice-subspace of V, contradicting our assumption.
Hence we may assume that $k \ge 2$ and the statement holds for spaces of dimension $k - 1$. Suppose L contains a nonzero vector $r \in \Lambda$. Let

$$L' = \mathrm{proj}_{\langle r\rangle^\perp}(L), \quad \Lambda' = \mathrm{proj}_{\langle r\rangle^\perp}(\Lambda).$$

By Lemma 5, Λ' is a lattice of $\langle r\rangle^\perp \cap V$. Also, L' is not a lattice subspace of $\langle r\rangle^\perp \cap V$ with respect to Λ', because if there exists a basis $a_1, \ldots, a_{k-1}$ of L' contained in Λ', then there exist scalars $\mu_1, \ldots, \mu_{k-1}$ such that $a_1 + \mu_1 r, \ldots, a_{k-1} + \mu_{k-1} r \in \Lambda$, but then $r, a_1 + \mu_1 r, \ldots, a_{k-1} + \mu_{k-1} r$ is a basis of L contained in Λ, a contradiction. By induction, there exists a point $y' \in \Lambda' \setminus L'$ at distance less than ε from L'. Since $y' \in \Lambda'$, there exists a scalar μ such that $y = y' + \mu r \in \Lambda$, and y has distance less than ε from L.
Thus $L \cap \Lambda = \{0\}$. By Lemma 7, there exists a nonzero vector $y \in \Lambda$ at distance less than ε from L. Since L does not contain any point in Λ other than the origin, $y \notin L$. □

2.2 Convex Analysis

We will define the basic terminology of convex analysis in the context of general linear (vector) spaces over the field $\mathbb{R}$. Consider any linear space V over $\mathbb{R}$. A set $A \subseteq V$ is *convex*, provided that for every $x, y \in A$, the *interval* $\{tx + (1-t)y \mid t \in (0,1)\}$ also belongs to A. Given two convex sets $A, B \subseteq V$, A is said to be a *relaxation* of B if $B \subseteq A$. A *convex combination* of points $a_1, \ldots, a_n \in V$ is a point $\sum_{i=1}^n t_i a_i$ such that $\sum_{i=1}^n t_i = 1$ and $t_i \geq 0$ for all i. The *convex hull* of a set $A \subseteq V$ is the smallest convex set containing A. We will denote the convex hull of A by $\mathrm{conv}(A)$. The following facts are well known and easy to verify.

Proposition 9. *Let $\mathcal{F}$ be a family of convex sets in V. Then $\cap_{F \in \mathcal{F}} F$ is convex.*

Proposition 10. *The convex hull of a set $A \subseteq V$ is the set of all convex combinations of points in A. That is,*

$$\mathrm{conv}(A) = \{t_1 x_1 + \ldots + t_n x_n \mid \textit{for every finite set of points } x_1, \ldots, x_n \in A, \textit{ and real numbers } t_1, \ldots, t_n \in (0,1) \textit{ with } \sum_{i=1}^n t_i = 1\}.$$

A nonempty set A of points in $\mathbb{R}^n$ is called a *cone* if $\lambda x + \mu y \in C$ for every $x, y \in A$ and $\lambda, \mu \geq 0$. A cone is convex by definition. For any convex set A, the set $\{y \in V \mid x + \lambda y \in A \text{ for all } x \in A, \lambda \geq 0\}$ is a cone and is called the *recession cone* of A. It will be denoted by $\mathrm{rec}(A)$. The set $\{y \in V \mid x + \lambda y \in A \text{ for all } x \in A, \lambda \in \mathbb{R}\}$ is a linear subspace of V and is called the *lineality space* of A. It will be denoted by $\mathrm{lin}(A)$. The following is a standard result in convex analysis (see for example Proposition 2.2.3 in [47]) .

Proposition 11. *A convex set A is unbounded if and only if* $\mathrm{rec}(A) \neq \{0\}$.

A *linear functional on V* is a map $f : V \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ and $f(tx) = tf(x)$ for all $x, y \in V$ and $t \in \mathbb{R}$. Let f be a linear functional on V, then the set $\{x \in V \mid f(x) = \alpha\}$ for some fixed $\alpha \in \mathbb{R}$ is called an *(affine) hyperplane*. The set $\{x \in V \mid f(x) \leq \alpha\}$ is called a *halfspace*.

A linear combination $t_1 a_1 + \ldots + t_n a_n$ with $\sum_{i=1}^n t_i = 1$ is called an *affine combination*. A set of points $a_1, \ldots, a_n$ are called *affinely independent* if whenever $t_1 a_1 + \ldots + t_n a_n$ with $\sum_{i=1}^n t_i = 0$, we must have $t_1 = t_2 = \ldots = t_n = 0$.

Given a set $A \subseteq V$, a point $v \in A$ is an *extreme point*, if it cannot be expressed as the convex combination of two points in A.

A function $p : V \to \mathbb{R}$ is called a *norm* provided that i) $p(v) \geq 0$ for all $v \in V$ and $p(v) = 0$ if and only if $u = 0$, ii) $p(tv) = |t|p(v)$ for all $v \in V$ and $t \in \mathbb{R}$, iii) $p(v_1) + p(v_2) \geq p(v_1 + v_2)$ for all $v_1, v_2 \in V$. A linear space V equipped with a norm is called a *normed linear space*. The norm defines an associated topology with it on the linear space V. The following separation theorem for normed linear spaces follows from the celebrated Hahn-Banach Theorem (see Barvinok [17], p. 119, Theorem 3.4).

Theorem 12. *Let V be a normed linear space. Let $A \subseteq V$ be a closed (with respect to the topology induced by the norm) convex set and let $u \notin A$. Then there exists a continuous linear functional and a real number α such that $f(x) \geq \alpha$ for all $x \in A$ and $f(u) < \alpha$.*

2.2.1 Convex Sets in $\mathbb{R}^n$

In this section we will be concerned with convex sets in $\mathbb{R}^n$. Let $C \subseteq \mathbb{R}^n$ be a closed convex set. Given a linear functional f and a real α, the set $H = \{x \in V \mid f(x) = \alpha\}$ is called a *supporting hyperplane* for C if the halfspace $\{x \in V \mid f(x) \leq \alpha\}$ contains C. The inequality $f(x) \leq \alpha\}$ is also called a *valid inequality* for C. The set $H \cap C$ is called a *face* if H is a supporting hyperplane for C. If $H \cap C$ is a single point, then it is called an *exposed point*. Clearly, exposed points are extreme. We will denote by $\mathrm{ext}(C)$ the set of extreme points and $\exp(C)$ the set of exposed points of C. We will need Straszewicz's theorem [59] (see Theorem 18.6 in [56]).

Theorem 13. *Given a closed convex set C, the set of exposed points of C is a dense subset of the set of extreme points of C.*

A function $f : \mathbb{R}^n \to \mathbb{R}$ is *positively homogeneous* if $f(tx) = tf(x)$ for every $x \in \mathbb{R}^n$ and $t \geq 0$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *convex* if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for all $x, y \in \mathbb{R}^n$ and $t \in (0, 1)$. A function that is convex and positively homogeneous is called *sublinear*. A function σ is *subadditive* if $\sigma(x_1 + x_2) \leq \sigma(x_1) + \sigma(x_2)$ for every $x_1, x_2 \in \mathbb{R}^n$. It is easy to show that σ is sublinear if and only if it is subadditive and positively homogeneous. Note that if σ is sublinear, then $\sigma(0) = 0$ (by positive homogeneity). The following proposition is also well-known in convex analysis [47].

Proposition 14. *Convex functions defined on all of $\mathbb{R}^n$ are continuous; so if f is sublinear and has domain $\mathbb{R}^n$ it is also continuous.*

For $K \subseteq \mathbb{R}^n$, we use the notation

$$K^* = \{y \in \mathbb{R}^n \,|\, \langle x, y\rangle \leq 1 \text{ for all } x \in K\} \tag{2.2}$$
$$\hat{K} = \{y \in K^* \,|\, \langle x, y\rangle = 1 \text{ for some } x \in K\}. \tag{2.3}$$

where $\langle x, y\rangle$ denotes the dot product of the two vectors x, y. The set K^* is called the *polar* of K. The set $\hat{K}$ is contained in the relative boundary of K^*. The polar K^* is a convex set whereas $\hat{K}$ is not convex in general.

The *support function* of a nonempty set $T \subset \mathbb{R}^n$ is defined by

$$\sigma_T(x) = \sup_{y \in T} \langle x, y\rangle \quad \text{for all } x \in \mathbb{R}^n.$$

It is straightforward to show that support functions are sublinear and $\sigma_T = \sigma_{\overline{\text{conv}}(T)}$ [47], where $\overline{\text{conv}}(T)$ is the closure of the convex hull of T.

The following result was obtained by the author in collaboration with Gerard Corneujols and Giacomo Zambelli.

Theorem 15. *Let $K \subset \mathbb{R}^n$ be a closed convex set containing the origin in its interior. If $\sigma : \mathbb{R}^n \to \mathbb{R}$ is a sublinear function such that $K = \{x \in \mathbb{R}^n \,|\, \sigma(x) \leq 1\}$, then $\sigma_{\hat{K}}(x) \leq \sigma(x)$ for all $x \in \mathbb{R}^n$.*

In the remainder we define $\rho_K = \sigma_{\hat{K}}$.

Let $K \subset \mathbb{R}^n$ be a closed convex set containing the origin in its interior. A standard concept in convex analysis [47, 56] is that of *gauge* (sometimes called Minkowski function), which is the function γ_K defined by

$$\gamma_K(x) = \inf\{t > 0 \,|\, t^{-1}x \in K\} \quad \text{for all } x \in \mathbb{R}^n.$$

By definition γ_K is nonnegative. One can readily verify that $K = \{x \,|\, \gamma_K(x) \leq 1\}$. It is well known that γ_K is the support function of K^* (see [47] Proposition 3.2.4).

Given any sublinear function σ such that $K = \{x \,|\, \sigma(x) \leq 1\}$, it follows from positive homogeneity that $\sigma(x) = \gamma_K(x)$ for every x where $\sigma(x) > 0$. Hence $\sigma(x) \leq \gamma_K(x)$ for all $x \in \mathbb{R}^n$. On the other hand we prove in Theorem 15 that the sublinear function ρ_K satisfies $\rho_K(x) \leq \sigma(x)$ for all $x \in \mathbb{R}^n$. Moreover, we will show below (Corollary 17) that $K = \{x \,|\, \rho_K(x) \leq 1\}$. In words, γ_K is the largest sublinear function such that $K = \{x \in \mathbb{R}^n \,|\, \sigma(x) \leq 1\}$ and ρ_K is the smallest.

Note that ρ_K can take negative values, so in general it is different from the gauge γ_K. Indeed the recession cone of K, which is the set $\text{rec}(K) = \{x \in$

$K \,|\, tx \in K$ for all $t > 0\}$, coincides with $\{x \in K \,|\, \sigma(x) \leq 0\}$ for every sublinear function σ such that $K = \{x \,|\, \sigma(x) \leq 1\}$. In particular $\rho_K(x)$ can be negative for $x \in \mathrm{rec}(K)$. For example, let $K = \{x \in \mathbb{R}^2 \,|\, x_1 \leq 1, x_2 \leq 1\}$. Then $K^* = \mathrm{conv}\{(0,0),(1,0),(0,1)\}$ and $\hat{K} = \mathrm{conv}\{(1,0),(0,1)\}$. Therefore, for every $x \in \mathbb{R}^2$, $\gamma_K(x) = \max\{0, x_1, x_2\}$ and $\rho_K(x) = \max\{x_1, x_2\}$. In particular, $\rho_K(x) < 0$ for every x such that $x_1 < 0$, $x_2 < 0$.

We now turn to the proof of Theorem 15. Let K be a closed convex set containing the origin in its interior. Let σ be a sublinear function such that $K = \{x \,|\, \sigma(x) \leq 1\}$. The boundary of K, denoted by $\mathbf{bd}(K)$, is the set $\{x \in K \,|\, \sigma(x) = 1\}$.

Lemma 16. *For every $x \notin \mathrm{rec}(K)$, $\sigma(x) = \rho_K(x) = \sup_{y \in K^*} \langle x, y \rangle$.*

Proof. Let $x \notin \mathrm{rec}(K)$. Then $t = \sigma(x) > 0$. By positive homogeneity, $\sigma(t^{-1}x) = 1$, hence $t^{-1}x \in \mathbf{bd}(K)$. Since K is closed and convex, there exists a supporting hyperplane for K containing $t^{-1}x$. Since $0 \in \mathbf{int}(K)$, this implies that there exists $\bar{y} \in K^*$ such that $(t^{-1}x)\bar{y} = 1$. In particular $\bar{y} \in \hat{K}$, hence by definition $\rho_K(x) \geq \langle x, \bar{y} \rangle = t$.

Furthermore, for any $y \in K^*$, $\langle t^{-1}x, y \rangle \leq 1$, hence $\langle x, y \rangle \leq t$, which implies $t \geq \sup_{y \in K^*} \langle x, y \rangle$. Thus

$$\rho_K(x) \geq t \geq \sup_{y \in K^*} \langle x, y \rangle \geq \sup_{y \in \hat{K}} \langle x, y \rangle = \rho_K(x),$$

where the last inequality holds since $\hat{K} \subset K^*$, hence equality holds throughout. □

Corollary 17. $K = \{x \,|\, \rho_K(x) \leq 1\}$.

Lemma 18. *Given an exposed point $\bar{y}$ of K^* different from the origin, there exists $x \in K$ such that $\langle x, \bar{y} \rangle = 1$ and $\langle x, y \rangle < 1$ for all $y \in K^*$ distinct from $\bar{y}$.*

Proof. If $\bar{y} \neq 0$ is an exposed point of K^*, then there exists a supporting hyperplane $H = \{y \,|\, \langle a, y \rangle = \beta\}$ such that $\langle a, \bar{y} \rangle = \beta$ and $\langle a, y \rangle < \beta$ for every $y \in K^* \setminus \{\bar{y}\}$. Since $0 \in K^*$ and $\bar{y} \neq 0$, $\beta > 0$. Thus the point $x = \beta^{-1}a \in K^{**} = K$ satisfies the statement. □

The next lemma states that $\hat{K}$ and $\hat{K} \cap \exp(K^*)$ have the same support function.

Lemma 19. *For every $x \in \mathbb{R}^n$, $\rho_K(x) = \sup_{y \in \hat{K} \cap \exp(K^*)} \langle x, y \rangle$.*

Proof. We first show that $\rho_K(x) = \sup_{y \in \hat{K} \cap \mathrm{ext}(K^*)} \langle x, y \rangle$. Given $y \in \hat{K}$ we show that there exists an extreme point y' of K^* in $\hat{K}$ such that $\langle x, y \rangle \leq \langle x, y' \rangle$. Since $y \in \hat{K}$, there exists $\bar{x} \in K$ such that $\langle \bar{x}, y \rangle = 1$. The point y is a convex combination of extreme points $y_1, \ldots, y_k$ of K^*, and each y_i satisfies $\langle \bar{x}, y^i \rangle = 1$. Thus $y^1, \ldots, y^k \in \hat{K}$, and $\langle x, y^i \rangle \geq \langle x, y \rangle$ for at least one i.

By Straszewicz's theorem (Theorem 13) the set of exposed points in K^* is a dense subset of the extreme points of K^*. By Lemma 18, all exposed points of K^* except the origin are in $\hat{K}$, hence $\exp(K^*) \cap \hat{K}$ is dense in $\mathrm{ext}(K^*) \cap \hat{K}$. Therefore $\rho_K(x) = \sup_{y \in \hat{K} \cap \exp(K^*)} \langle x, y \rangle$. □

Proof of Theorem 15. By Lemma 16, we only need to show $\sigma(x) \geq \rho_K(x)$ for points $x \in \mathrm{rec}(K)$. By Lemma 19 it is sufficient to show that, for every exposed point $\bar{y}$ of K^* contained in $\hat{K}$, $\sigma(x) \geq \langle x, \bar{y} \rangle$.

Let $\bar{y}$ be an exposed point of K^* in $\hat{K}$. By Lemma 18 there exists $\bar{x} \in K$ such that $\langle \bar{x}, \bar{y} \rangle = 1$ and $\langle \bar{x}, y \rangle < 1$ for all $y \in K^*$ distinct from $\bar{y}$. Note that $\bar{x} \in \mathbf{bd}(K)$.

We observe that for all $\delta > 0$, $\bar{x} - \delta^{-1} x \notin \mathrm{rec}(K)$. Indeed, since $x \in \mathrm{rec}(K)$, $\bar{x} + \delta^{-1} x \in K$. Hence $\bar{x} - \delta^{-1} x \notin \mathbf{int}(K)$ because $\bar{x} \in \mathbf{bd}(K)$. Since $0 \in \mathbf{int}(K)$ and $\bar{x} - \delta^{-1} x \notin \mathbf{int}(K)$, then $\bar{x} - \delta^{-1} x \notin \mathrm{rec}(K)$. Thus by Lemma 16

$$\sigma(\bar{x} - \delta^{-1} x) = \sup_{y \in K^*} \langle \bar{x} - \delta^{-1} x, y \rangle. \tag{2.4}$$

Since $\bar{x} \in \mathbf{bd}(K)$, $\sigma(\bar{x}) = 1$. By subadditivity, $1 = \sigma(\bar{x}) \leq \sigma(\bar{x} - \delta^{-1} x) + \sigma(\delta^{-1} x)$. By positive homogeneity, the latter implies that $\sigma(x) \geq \delta - \delta \sigma(\bar{x} - \delta^{-1} x)$ for all $\delta > 0$. By (2.4),

$$\sigma(x) \geq \inf_{y \in K^*} [\delta(1 - \langle \bar{x}, y \rangle + \langle x, y \rangle],$$

hence

$$\sigma(x) \geq \sup_{\delta > 0} \inf_{y \in K^*} [\delta(1 - \langle \bar{x}, y \rangle + \langle x, y \rangle].$$

Let $g(\delta) = \inf_{y \in K^*} \delta(1 - \langle \bar{x}, y \rangle + \langle x, y \rangle$. Since $\bar{x} \in K$, $1 - \langle \bar{x}, y \rangle \geq 0$ for every $y \in K^*$. Hence $\delta(1 - \langle \bar{x}, y \rangle + \langle x, y \rangle$ defines an increasing affine function of δ for each $y \in K^*$, therefore $g(\delta)$ is increasing and concave. Thus $\sup_{\delta > 0} g(\delta) = \lim_{\delta \to \infty} g(\delta)$.

Since K^* is compact, for every $\delta > 0$ there exists $y(\delta) \in K^*$ such that $g(\delta) = \delta(1 - \langle \bar{x}, y(\delta) \rangle) + \langle x, y(\delta) \rangle$. Furthermore there exists a sequence $(\delta_i)_{i \in \mathbb{N}}$ such that $\lim_{i \to \infty} \delta_i = +\infty$ and the sequence $(y_i)_{i \in \mathbb{N}}$ defined by $y_i = y(\delta_i)$ converges, because

in a compact set every sequence has a convergent subsequence. Let $y^* = \lim_{i\to\infty} y_i$.

We conclude the proof by showing that $\sigma(x) \geq \langle x, y^* \rangle$ and $y^* = \bar{y}$.

$$\begin{aligned}
\sigma(x) \geq \sup_{\delta>0} g(\delta) &= \lim_{i\to\infty} g(\delta_i) \\
&= \lim_{i\to\infty} [\delta_i(1 - \langle \bar{x}, y_i \rangle) + \langle x, y_i \rangle] \\
&= \lim_{i\to\infty} \delta_i(1 - \langle \bar{x}, y_i \rangle) + \langle x, y^* \rangle \\
&\geq \langle x, y^* \rangle
\end{aligned}$$

where the last inequality follows from the fact that $\delta_i(1 - \langle \bar{x}, y_i \rangle) \geq 0$ for all $i \in \mathbb{N}$. Finally, since $\lim_{i\to\infty} \delta_i(1 - \langle \bar{x}, y_i \rangle)$ is bounded and $\lim_{i\to\infty} \delta_i = +\infty$, it follows that $\lim_{i\to\infty}(1 - \langle \bar{x}, y_i \rangle) = 0$, hence $\langle \bar{x}, y^* \rangle = 1$. By our choice of $\bar{x}$, $\langle \bar{x}, y \rangle < 1$ for every $y \in K^*$ distinct from $\bar{y}$. Hence $y^* = \bar{y}$. □

2.3 Polyhedral Theory

All the results presented in this section can be found in Schrijver's text [58]. We will use the following formulation of Farkas' Lemma in this thesis.

Lemma 20. *Consider linear system*

$$\begin{aligned}
Ax &\leq b \\
Cx &= d
\end{aligned}$$

where A is an $m \times n$ matrix, C is an $p \times n$ matrix, $b \in \mathbb{R}^m$ and $d \in \mathbb{R}^p$. This system is infeasible if and only if there exist $\gamma \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}^p$ such that $\gamma \geq 0$, $\gamma^T A + \lambda^T C = 0$ and $\gamma^T b + \lambda^T d < 0$.

A cone C is *polyhedral* if $C = \{x \mid Ax \leq 0\}$ for some matrix A. A cone is said to be *finitely generated* by a set of vectors $x_1, \ldots, x_m \in \mathbb{R}^n$ if $C = \{\lambda_1 x_1 + \ldots + \lambda_m x_m \mid \lambda_1, \ldots, \lambda_m \geq 0\}$. It is a consequence of Farkas' Lemma that

Theorem 21. *A cone is polyhedral if and only if it is finitely generated.*

A set P in $\mathbb{R}^n$ is a (convex) polyhedron if $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ for some matrix A and vector b, i.e. if P is the intersection of a finite number of affine halfspaces. A (convex) polytope is the convex hull of a finite number of points in $\mathbb{R}^n$.

The Minkowski sum of two sets X, Y in $\mathbb{R}^n$ is denoted by $X+Y = \{w \in \mathbb{R}^n \mid w = x+y$ for some $x \in X, y \in Y\}$. The following decomposition theorem for polyhedra is also known as the Minkowski-Weyl Theorem.

Theorem 22. *A set P of points in $\mathbb{R}^n$ is a polyhedron if and only if $P = Q + C$ for some polytope P and some polyhedral cone C.*

Moreover, the polyhedral cone C in the above decomposition is given by $\{x \in \mathbb{R}^n \mid Ax \leq 0\}$ and is the recession cone of P. The lineality space of P is given by the set $\{x \in \mathbb{R}^n \mid Ax = 0\}$.

The *dimension* of P is equal to the maximum number of affinely independent points in P. A *facet* of P is a maximal face (as defined in Section 2.2.1) distinct from P (maximal with respect to inclusion amongst faces of P). The dimension of a facet of P is one less than the dimension of P.

We will also refer to the following theorem of Meyer [54]. Consider any polyhedron P. We denote by P_I the convex hull of all integer points inside P, i.e. $P_I = \mathrm{conv}(\{x \in P \mid \text{all coordinates of } x \text{ are integral}\})$. A polyhedron $P = \{x \mid Ax \leq b\}$ is *rational* if the matrix A and the vector b have rational entries.

Theorem 23. *For any rational polyhedron P, the set P_I is a polyhderon again. If P_I is nonempty, then* $\mathrm{rec}(P) = \mathrm{rec}(P_I)$.

2.4 Corner Polyhedron and Intersection Cuts

2.4.1 Corner Polyhedron

Corner polyhedra were introduced by Gomory [43] as a generalization of his original cutting plane method [42] and this structure was studied extensively by Gomory and Johnson [44, 48]. We introduce this concept and the notation in this section.

Consider a general MILP in the following standard form. Let A be a rational $m \times n$ matrix, b be a rational $m \times 1$ column vector. We are interested in $x \in \mathbb{R}^n$ such that

$$\begin{aligned} Ax &= b, \\ x &\geq 0, \quad x_j \in \mathbb{Z} \text{ for } j \in I \subseteq \{1, \ldots n\} \end{aligned} \tag{2.5}$$

We assume, without loss of generality, that A has full row rank. A basis D is a maximal linearly independent set of columns in A; so D is an invertible $m \times$

m matrix. Corresponding to any basis D, we define the corner polyhedron with respect to D, denoted by $corner(D)$ as the convex hull of all $x \in \mathbb{R}^n$ such that

$$\begin{array}{rcl} x_{\mathcal{D}} & = & D^{-1}b - D^{-1}Nx_{\mathcal{N}}, \\ x_{\mathcal{N}} & \geq & 0, \quad x_j \in \mathbb{Z} \text{ for } j \in I \subseteq \{1, \ldots n\} \end{array} \tag{2.6}$$

where $x_{\mathcal{D}}$ refers to the variables corresponding to the columns of D (a.k.a the basic variables) and $x_{\mathcal{N}}$ refers to the remaining variables (a.k.a the non-basic variables). To simplify notation, we will denote the set of indices of the basic variables by $\mathcal{D}$, and the non-basic variables by $\mathcal{N}$. The point $\bar{x} = (D^{-1}b, 0)$ is called the *basic solution* corresponding to the basis D. Let $LP(corner(D))$ be the *relaxation* of $corner(D)$ where the integrality constraints are dropped on all the variables in (2.6). It is well-known that $corner(D) \neq LP(corner(D))$ if and only if $\bar{x}$ does not satisfy the integrality constraints. Furthermore, $corner(D)$ is a *relaxation* of (2.5) because the non-negativity constraints on the basic variables have been dropped. This implies that valid inequalities for $corner(D)$ are also valid inequalities for (2.5).

The significance of this corner polyhedron lies in the fact that most cutting planes for general MILPs (e.g. GMI cuts, Split cuts, Reduce-and-Split cuts, MIR cuts and so forth) are obtained by solving the linear relaxation using linear programming methods and then using integrality arguments to find an inequality which cuts off the fractional solution of the linear relaxation. Many families of such cutting planes, including the four examples listed above, are valid inequalities for $corner(D)$ for some appropriate basis D, which cut off the corresponding basic solution $\bar{x}$. D usually corresponds to the optimal basis in the standard simplex method for solving the LP relaxation. Moreover, for the derivation of these cutting planes, one needs to only consider a relaxation of the corner polyhedra obtained by retaining only one equality constraint. Therefore, having a characterization of polyhedra in $\mathbb{R}^n$ defined using constraints like (2.6) would be an automatic generalization of most cutting planes used in practice. **In addition, corner polyhedra have facets which cannot be obtained by applying integrality arguments to a single constraint from the system [8]**. So the analysis of corner polyhedra provides us with new families of cutting planes which may be useful for solving general MILPs.

2.4.2 Intersection Cuts

Balas introduced the concept of Intersection Cuts in [11] to obtain valid inequalities for the corner polyhedron (2.6). Note that in the relaxation (2.6), we can drop the constraints which correspond to continuous basic variables, as these appear in only one equation and no other constraint. Moreover, for simplicity of notation, let us reorder the indices of the variables such that the integer constrained variables are indexed by $\{1, \ldots, p\}$ and so $\mathcal{D} \subseteq \{1, \ldots, p\}$. Balas gave the following procedure to derive valid inequalities for $corner(D)$ which cut off the fractional solution $\bar{x}$ (when $corner(D) \neq LP(corner(D))$).

We first set up some notation for describing Balas' procedure. $LP(corner(D))$ is a translated cone with the unique vertex $\bar{x}$, i.e. $LP(corner(D) = \bar{x} + C$ where C is a polyhedral cone. C is generated by $|\mathcal{N}|$ rays r^i, $i \in \mathcal{N}$ (recall that $\mathcal{N}$ is the set of indices for the non-basic variables) which are given by

$$r_h^i = \begin{cases} (D^{-1}N_i)_h & h \in D \\ 1 & h = i \\ 0 & h \in N \setminus \{i\} \end{cases}$$

where N_i denotes the column of N indexed by i. To generate a valid inequality to cut $\bar{x}$, Balas considers a closed convex body B in $\mathbb{R}^n$ such that it contains the basic solution $\bar{x}$ in its interior, but does not contain any point from $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ in its interior. Define

$$a_j = \inf\{t > 0 \,|\, \frac{r^j}{t} \in B\}$$

for every $j \in \mathcal{N}$ and consider the inequality

$$\sum_{j \in \mathcal{N}} a_j x_j \geq 1 \tag{2.7}$$

Balas showed that (2.7) is valid for $corner(D)$ and clearly cuts off the point $\bar{x}$. This cut is called the *intersection cut* obtained from B. Conforti, Cornuejols and Zambelli have also shown that the converse holds and that every valid inequality for $corner(D)$ comes from an intersection cut corresponding to some convex body satisfying the above conditions [27]. Note that one can view the above inequality as $\sum_{j \in \mathcal{N}} \gamma_B(r^j) x_j \geq 1$, where γ_B is the gauge function for the convex set B. This connection will be explored further in this thesis.

2.5 Variants of the Corner Polyhedron

Many different relaxations and variants of the corner polyhedron have been investigated with the purpose of obtaining cutting planes of general MILPs. The central role of the corner polyhedron was outlined in Section 2.4.1. We now discuss some other variants studied in the literature. This serves the dual purpose of introducing notation which will be useful later.

2.5.1 Infinite Group Problem

Gomory and Johnson investigate the special case of the corner polyhedron when all the non-basic variables are integer constrained. In this context they develop the *group problem* and analyze various aspects of this problem in the context of Mixed Integer Linear Programming. See [43, 44, 48, 46]. We outline the basic ideas in this section. We consider the system

$$\begin{array}{rcl} x_{\mathcal{D}} & = & D^{-1}b - D^{-1}Nx_{\mathcal{N}}, \\ x_{\mathcal{N}} & \geq & 0, \quad x_j \in \mathbb{Z} \text{ for } j \in \{1, \ldots n\} \end{array} \tag{2.8}$$

Note that the difference with (2.6) is that all the variables are now constrained to be integer. Then one can rewrite the above system in the form

$$\begin{array}{c} D^{-1}b \equiv D^{-1}Nx_{\mathcal{N}} \pmod 1, \\ x_i \in \mathbb{Z}_+ \text{ for all } i \in N \end{array} \tag{2.9}$$

If one views the vectors in $\mathbb{R}^{|B|}$ as a group (with addition modulo 1 as the group operation), then the above system can be interpreted in terms of a group equation. In general, we can consider an Abelian group G and analyze the convex hull of all solutions to the equation

$$\begin{array}{c} g_0 \equiv \sum_{i=1}^{n} g_i x_i, \\ x_i \in \mathbb{Z}_+ \text{ for all } i \end{array} \tag{2.10}$$

where $g_0, g_1, \ldots, g_n$ are elements from the group G. This is known as the group problem with group G and right hand side $g_0 \in G$. The convex hull of solutions to (2.10) is denoted by $P(G, g_0)$. This concept was introduced by Gomory in [43] to study the corner polyhedra and he considered the case when G is a finite Abelian

group, since any corner polyhedron will have columns $D^{-1}N_i$ which are elements from a finite group of order at most $|\det D|^k$, where k is the number of basic variables.

Gomory and Johnson studied the case when $G = [0,1]^k$ in [44] and [46]. The motivation is that for any finite subgroup G' of $[0,1]^k$, the group problem $P(G', f)$ with right hand side $f \in [0,1]^k$ is a face of $P([0,1]^k, f)$ and hence any corner polyhedron (with the same right hand side) becomes a face of this *master polyhedron* $P([0,1]^k, f)$. This problem is sometimes referred to as the *infinite group problem* in the literature. Formally, we study the following system.

$$\begin{array}{rcl} \sum_{r\in[0,1]^k} r s_r & = & f \\ s_r & \in & \mathbb{Z}_+ \qquad \forall r \in [0,1]^k \\ & & s \text{ has finite support,} \end{array} \tag{2.11}$$

This abstraction has the following two advantages :

1. We have only one parameter f, the right hand side, as compared to the corner polyhedron, where we also need to keep track of the specific columns.

2. More significantly, an appropriate notion of a valid inequality can be defined for this infinite group problem, which corresponds to a function $\psi : [0,1]^k \to \mathbb{R}$. This gives a "formula" to obtain the coefficients of a valid inequality for *any* given finite problem. One simply plugs in the appropriate non-basic column from $[0,1]^k$ into ψ and gets the coefficient for the corresponding non-basic variable in a valid inequality.

This idea of abstraction to an infinite group to obtain "formulas" for all finite problems will be a central theme in this thesis.

2.5.2 Continuous Relaxation

Andersen et al. [8] introduced a relaxation of the corner polyhedron, where the integrality conditions on the non-basic variables are dropped. We can formulate this relaxation in the following notation, which provides a little more geometric intuition.

$$\begin{array}{rcl} x & = & f + \sum_{i=1}^{k} r^i s_i \\ x & \in & \mathbb{Z}^m \\ s_i & \geq & 0 \quad \text{for all } i \end{array} \tag{2.12}$$

In the above, $f \in \mathbb{R}^m$ corresponds to $D^{-1}b$ in (2.6) and r^i's are rays in $\mathbb{R}^m$ which correspond to the non-basic columns $(-D^{-1}N_i)$ in (2.6) (N_i denotes the i-th column of the matrix N). x corresponds to $x_{\mathcal{D}}$ and s corresponds to $x_{\mathcal{N}}$. Note that to describe the solutions of (2.12), one only needs to records the values of the s_i variables. We use $R_f(r^1, \ldots, r^k)$ to denote the set of all points s such that (2.12) is satisfied.

Since $\mathrm{conv}(R_f(r^1, \ldots, r^k))$ is a relaxation of the corner polyhedron, any valid inequality for $\mathrm{conv}(R_f(r^1, \ldots, r^k))$ is valid for the corner polyhedron and hence for our initial MILP problem from which the corner polyhedron is derived. For the same motivations as discussed in the previous section, it is also fruitful to study the following problem, with an infinite number of variables, first studied by Borozan and Cornuejols [25].

$$\begin{array}{rl} x = & f + \displaystyle\sum_{r \in \mathbb{R}^n} r s_r \\ & x \in \mathbb{Z}^m, \\ & s_r \in \mathbb{R}_+, \ \forall r \in \mathbb{R}^n, \\ & s \text{ has finite support.} \end{array} \tag{2.13}$$

As outlined in the previous section, the main motivation behind this model is to obtain functions or "formulas" which give coefficients for valid inequalities for *any* finite problem with the same right hand side f. Moreover, Borozan and Cornuejols [25] showed a one-to-one correspondence between these functions and maximal lattice-free convex sets, which are a specific kind of convex sets. This connection will be explored further in Chapter 3. This equivalence of valid inequalities and geometric objects leads one to obtain various results about cutting planes using geometric intuition and arguments. This will be illustrated in the subsequent chapters of this thesis.

Chapter 3

Maximal Lattice-free Convex Sets and Minimal Valid Inequalities

This chapter deals with the connections between the valid inequalities for integer programs and certain geometric objects termed as maximal lattice-free convex sets. Many different results will be proved with the following general principle behind all of them.

> *There is a one-to-one correspondence between "non-dominated" valid inequalities for integer programs and maximal (with respect to inclusion) convex sets with the property that their interior does not contain any feasible integer solutions.*

For this purpose, we develop the theory of the so-called maximal lattice-free convex sets. The study of maximal lattice-free convex sets dates back to Minkowski's work on the geometry of numbers. Connections between integer programming and the geometry of numbers were investigated in the 1980s starting with the work of Lenstra [50]. See Lovász [52] for a survey. Recent work in cutting plane theory [6],[7],[9],[8][18],[25],[34],[37],[38],[40],[60] has generated renewed interest in the study of maximal lattice-free convex sets. As pointed out in Section 2.4.2, the relation between lattice-free convex sets and valid inequalities in integer programming was first observed by Balas [11].

This chapter is divided into two main sections. The first section (Section 3.1) develops the theory of maximal lattice-free convex sets and establishes the precise connection between these convex sets and valid inequalities. The second section

(Section 3.2) generalizes the concept of lattice-free convex sets to the so-called S-free convex sets and the correspondence with another class of valid inequalities is shown. This work was done by the author in collaboration with Michele Conforti, Gerard Cornuejols and Giacomo Zambelli and it appears in the two papers [21, 22].

3.1 Maximal Lattice-free Convex Sets in Linear Subspaces of $\mathbb{R}^n$

Let W be an affine subspace of $\mathbb{R}^n$. Assume that $W \cap \mathbb{Z}^n \neq \emptyset$. We say that a set $B \subset \mathbb{R}^n$ is a *maximal lattice-free convex set in* W if $B \subset W$, B is convex, has no integer point in its relative interior, and it is inclusionwise maximal with these three properties. The subspace W is said to be *rational* if it is generated by the integer points in W. So, if we denote by V the affine hull of the integer points in W, $V = W$ if and only if W is rational. If W is not rational, then the inclusion $V \subset W$ is strict. When W is not rational, we will also say that W is *irrational*.

Theorem 24. *Let $W \subset \mathbb{R}^n$ be an affine space containing an integral point and V the affine hull of $W \cap \mathbb{Z}^n$. A set $S \subset W$ is a maximal lattice-free convex set of W if and only if one of the following holds:*

(i) S is a polyhedron in W whose dimension equals $\dim(W)$, $S \cap V$ is a maximal lattice-free convex set of V whose dimension equals $\dim(V)$, and for every facet F of S, $F \cap V$ is a facet of $S \cap V$;

(ii) S is an affine hyperplane of W such that $S \cap V$ is an irrational hyperplane of V;

(iii) S is a half-space of W that contains V on its boundary.

A characterization of maximal lattice-free convex sets of V, needed in (i) of the previous theorem, is given by the following.

Theorem 25. (Lovász [52]) *Let V be a rational affine subspace of $\mathbb{R}^n$ containing an integral point. A set $S \subset V$ is a maximal lattice-free convex set of V if and only if one of the following holds:*

(i) S is a polyhedron of the form $S = P + L$ where P is a polytope, L is a rational linear space, $\dim(S) = \dim(P) + \dim(L) = \dim(V)$, S does not contain any

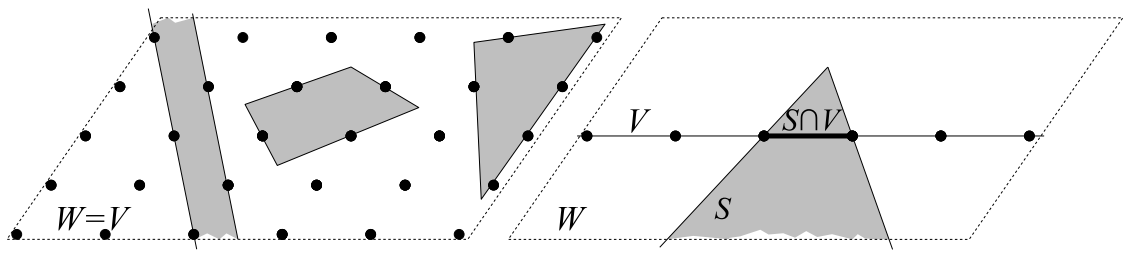


Figure 3.1: Maximal lattice-free convex sets in a 2-dimensional subspace (Theorem 24(i)).

integral point in its interior and there is an integral point in the relative interior of each facet of S;

(ii) S is an irrational affine hyperplane of V.

The polyhedron $S = P + L$ in Theorem 25(i) is called a *cylinder over the polytope* P and can be shown to have at most $2^{\dim(P)}$ facets [39].

Theorem 24 is new and it is used in the proof of our main result, Theorem 26. It is also used to prove the last theorem in [25]. Theorem 25 is due to Lovász ([52] Proposition 3.1). Lovász only gives a sketch of the proof and it is not clear how case (ii) in the above theorem arises in his sketch or in the statement of his proposition. Therefore in Section 3.1.1 we will prove both theorems for the sake of completeness.

Figure 3.1 shows examples of maximal lattice free convex sets in a 2-dimensional affine subspace W of $\mathbb{R}^3$. We denote by V the affine space generated by $W \cap \mathbb{Z}^3$. In the first picture W is rational, so $V = W$, while in the second one V is a subspace of W of dimension 1.

In the second part of this section, we show a connection between maximal lattice-free convex sets in affine subspaces and mixed-integer linear programming. The relaxation (2.12) was studied by Andersen et al. [8] in the case of two rows and Borozan and Cornuéjols [25] for the general case. Recall that we denote the convex hull of solutions to (2.12) by $\mathrm{conv}(R_f(r^1, \ldots, r^k))$. In these papers they showed that the irredundant valid inequalities for $\mathrm{conv}(R_f(r^1, \ldots, r^k))$ correspond to maximal lattice-free convex sets in $\mathbb{R}^m$. In [8, 25] data are assumed to be rational. Here we consider the case where $f, r^1, \ldots, r^k$ may have irrational entries.

As mentioned in Section 2.5.2, Borozan and Cornuejols studied the model (2.13) motivated by the same reasons as Gomory and Johnson were for studying the infinite group problem (2.11). We discuss these motivations now in a little more detail. Note that every valid inequality for $\mathrm{conv}(R_f(r^1, \ldots, r^k))$ be written in the form $\sum_{j=1}^m \alpha_j s_j \geq \alpha$. We are interested in "formulas" for deriving such inequalities. More formally, we are interested in functions $\psi : \mathbb{R}^m \to \mathbb{R}$ and a real number α such that the inequality

$$\sum_{j=1}^k \psi(r^j) s_j \geq \alpha$$

is valid for $\mathrm{conv}(R_f(r^1, \ldots, r^k))$ for every choice of k and vectors $r^1, \ldots, r^k \in \mathbb{R}^m$. We refer to such functions ψ as *valid functions* (with respect to f). This further simplifies the analysis since the only parameter in the problem is f. We next formally define the notions of these valid functions and the corresponding valid inequalities for $\mathrm{conv}(R_f(r^1, \ldots, r^k))$.

Let W be any linear subspace of $\mathbb{R}^m$. Given a point $f \in \mathbb{R}^m$, let V be the affine hull of $(f+W) \cap \mathbb{Z}^q$. Notice that $f+W$ and V coincide if and only if W is a rational space. We next introduce a slight generalization of the Borozan and Cornuéjols model (2.13). Let $R_f(W)$ be the set of points $s = (s_r)_{r \in W}$ of $\mathbb{R}^W$ satisfying

$$\begin{aligned} & f + \sum_{r \in W} r s_r \in \mathbb{Z}^q \\ & s_r \geq 0, \quad r \in W \\ & s \in \mathcal{W} \end{aligned} \tag{3.1}$$

where $\mathcal{W}$ is the set of all $s \in \mathbb{R}^W$ with *finite support*, i.e. the set $\{r \in W \mid s_r > 0\}$ has finite cardinality. We wish to consider linear subspaces of $\mathbb{R}^n$ in the definition, because given a finite problem (2.12), the span of the vectors $r^1, \ldots, r^k$ may not be all of $\mathbb{R}^n$. Notice that $\mathrm{conv}(R_f(r^1, \ldots, r^k)) = R_f(W) \cap \{s \in \mathcal{W} \mid s_r = 0 \text{ for all } r \neq r^1, \ldots, r^k\}$, where $W = \langle r^1, \ldots, r^k \rangle$ is the linear space generated by $r^1, \ldots, r^k$.

Given a function $\psi : W \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, the linear inequality

$$\sum_{r \in W} \psi(r) s_r \geq \alpha \tag{3.2}$$

is *valid for* $R_f(W)$ if it is satisfied by every $s \in R_f(W)$.

Note that, given a valid inequality (3.2) for $R_f(W)$, the inequality

$$\sum_{j=1}^k \psi(r^j) s_j \geq \alpha$$

is valid for $\mathrm{conv}(R_f(r^1, \ldots, r^k))$. Hence a characterization of valid linear inequalities for $R_f(W)$ provides a characterization of valid linear inequalities for $\mathrm{conv}(R_f(r^1, \ldots, r^k))$. This is the formal notion behind the idea of using valid functions as generic formulas for every possible finite problem.

Next we observe how maximal lattice-free convex sets in $f + W$ give valid linear inequalities for $R_f(W)$. Let B be a maximal lattice-free convex set in $f + W$ containing f in its interior. Since, by Theorem 24, B is a polyhedron and since f is in its interior, there exist $a_1, \ldots, a_t \in \mathbb{R}^q$ such that $B = \{x \in f + W \mid a_i(x - f) \leq 1,\ i = 1 \ldots, t\}$. We define the function $\psi_B : W \to \mathbb{R}$ by

$$\psi_B(r) = \max_{i=1,\ldots,t} a_i r.$$

Note that the function ψ_B is *subadditive*, i.e. $\psi_B(r) + \psi_B(r') \geq \psi_B(r + r')$, and *positively homogeneous*, i.e. $\psi_B(\lambda r) = \lambda \psi_B(r)$ for every $\lambda \geq 0$. We claim that

$$\sum_{r \in W} \psi_B(r) s_r \geq 1$$

is valid for $R_f(W)$.

Indeed, let $s \in R_f(W)$, and $x = f + \sum_{r \in W} r s_r$. Note that $x \in \mathbb{Z}^n$, thus $x \notin \mathbf{int}(B)$. Then

$$\sum_{r \in W} \psi_B(r) s_r = \sum_{r \in W} \psi_B(r s_r) \geq \psi_B(\sum_{r \in W} r s_r) = \psi_B(x - f) \geq 1,$$

where the first equation follows from positive homogeneity, the first inequality follows from subadditivity of ψ_B and the last one follows from the fact that $x \notin \mathbf{int}(B)$.

We will show that all nontrivial irredundant valid linear inequalities for $R_f(W)$ are indeed of the type described above. Furthermore, if W is irrational, we will see that $R_f(W)$ is contained in a proper affine subspace of $\mathcal{W}$, so each inequality has infinitely many equivalent forms. Note that, by definition of ψ_B, $\psi_B(r) > 0$ if r is not in the recession cone of B, $\psi_B(r) < 0$ when r is in the interior of the recession cone of B, while $\psi_B(r) = 0$ when r is on the boundary of the recession cone of B. We will show that one can always choose a form of the inequality so that ψ_B is a nonnegative function. We make this more precise in the next theorem.

Given a point $s \in R_f(W)$, then $f + \sum_{r \in W} r s_r \in \mathbb{Z}^q \cap (f + W)$. Thus $\mathrm{conv}(R_f(W))$

is contained in the affine subspace $\mathcal{V}$ of $\mathcal{W}$ defined as

$$\mathcal{V} = \{s \in \mathcal{W} \mid f + \sum_{r \in W} r s_r \in V\}.$$

Observe that, given $C \in \mathbb{R}^{\ell \times q}$ and $d \in \mathbb{R}^{\ell}$ such that $V = \{x \in f + W \mid Cx = d\}$, we have

$$\mathcal{V} = \{s \in \mathcal{W} \mid \sum_{r \in W} (Cr) s_r = d - Cf\}. \tag{3.3}$$

A linear inequality $\sum_{r \in W} \psi(r) s_r \geq \alpha$ that is satisfied by every element in $\{s \in \mathcal{V} \mid s_r \geq 0$ for every $r \in W\}$ is said to be *trivial*.

We say that inequality $\sum_{r \in W} \psi(r) s_r \geq \alpha$ *dominates* inequality $\sum_{r \in W} \psi'(r) s_r \geq \alpha$ if $\psi(r) \leq \psi'(r)$ for all $r \in W$. Note that, for any $\bar{s} \in \mathcal{W}$ such that $\bar{s}_r \geq 0$ for all $r \in W$, if $\bar{s}$ satisfies the first inequality, then $\bar{s}$ also satisfies the second. A valid inequality $\sum_{r \in W} \psi(r) s_r \geq \alpha$ for $R_f(W)$ is *minimal* if it is not dominated by any valid linear inequality $\sum_{r \in W} \psi'(r) s_r \geq \alpha$ for $R_f(W)$ such that $\psi' \neq \psi$. It is not obvious that nontrivial valid linear inequalities are dominated by minimal ones. We will show that this is the case. Note that it is not even obvious that minimal valid linear inequalities exist.

We will show that, for any maximal lattice-free convex set B of $f + W$ with f in its interior, the inequality $\sum_{r \in W} \psi_B(r) s_r \geq 1$ is a minimal valid inequality for $R_f(W)$. The main result of this section is a converse, stated in the next theorem.

Given two valid inequalities $\sum_{r \in W} \psi(r) s_r \geq \alpha$ and $\sum_{r \in W} \psi'(r) s_r \geq \alpha'$ for R_f, we say that they are *equivalent* if there exist $\rho > 0$ and $\lambda \in \mathbb{R}^{\ell}$ such that $\psi(r) = \rho \psi'(r) + \lambda^T Cr$ and $\alpha = \rho \alpha' + \lambda^T (d - Cf)$.

Theorem 26. *Every nontrivial valid linear inequality for $R_f(W)$ is dominated by a nontrivial minimal valid linear inequality for $R_f(W)$.*
Every nontrivial minimal valid linear inequality for $R_f(W)$ is equivalent to an inequality of the form

$$\sum_{r \in W} \psi_B(r) s_r \geq 1$$

such that $\psi_B(r) \geq 0$ for all $r \in W$ and B is a maximal lattice-free convex set in $f + W$ with f in its interior.

This theorem generalizes earlier results about the case when W is a rational space (Borozan and Cornuéjols [25]). In their paper, they build on Balas' intersection cut idea, where the gauge function of a convex set is used to derive the

coefficients of the inequality as observed in Section 2.4.2. However the proof for irrational spaces is much more complicated. In the rational case it is immediate that all valid linear inequalities are of the form $\sum_{r\in W}\psi(r)s_r \geq 1$ with ψ nonnegative. From this, it follows easily that ψ must be equal to ψ_B for some maximal lattice-free convex set B using the gauge of B. In the irrational case, valid linear inequalities might have negative coefficients. For minimal inequalities, however, Theorem 26 shows that there always exists an equivalent one where all coefficients are nonnegative. The function ψ_B is nonnegative if and only if the recession cone of B has empty interior. Although there are nontrivial minimal valid linear inequalities arising from maximal lattice-free convex sets whose recession cone is full dimensional, Theorem 26 states that there always exists a maximal lattice-free convex set whose recession cone is not full dimensional that gives an equivalent inequality. A crucial ingredient in showing this is a new result about sublinear functions (Theorem 15), which defines an analogous concept to the gauge of convex set that is more suited in the current setting.

In light of Theorem 26, it is a natural question to ask what is the subset of $\mathcal{W}$ obtained by intersecting the set of nonnegative elements of $\mathcal{V}$ with all half-spaces defined by inequalities $\sum_{r\in W}\psi(r)s_r \geq 1$ as in Theorem 26. In a finite dimensional space, the intersection of all half-spaces containing a given convex set C is the closure of C. Things are more complicated in infinite dimension. First of all, while in finite dimension all norms are topologically equivalent, and thus the concept of closure does not depend on the choice of a specific norm, in infinite dimension different norms may produce different topologies. Secondly, in finite dimensional spaces linear functions are always continuous, while in infinite dimension there always exist linear functions that are not continuous. In particular, half-spaces (i.e. sets of points satisfying a linear inequality) are not always closed in infinite dimensional spaces (see Conway [30] for example).

To illustrate this, note that if $\mathcal{W}$ is endowed with the Euclidean norm, then $\mathbf{0} = (0)_{r\in W}$ belongs to the closure of $\mathrm{conv}(R_f(W))$ with respect to this norm, as shown next. Let $\bar{x}$ be an integral point in $f + W$ and let $\bar{s}$ be defined by

$$\bar{s}_r = \begin{cases} \frac{1}{k} & \text{if } r = k(\bar{x} - f), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, for every choice of k, $\bar{s} \in R_f(W)$, and for k that goes to infinity the point $\bar{s}$ is arbitrarily close to $\mathbf{0}$ with respect to the Euclidean distance. Now, given a valid linear inequality $\sum_{r\in W}\psi(r)s_r \geq 1$ for $\mathrm{conv}(R_f(W))$, since $\sum_{r\in W}\psi(r)0 = 0$ the hyperplane $\mathcal{H} = \{s \in \mathcal{W} : \sum_{r\in W}\psi(r)s_r = 1\}$ separates strictly $\mathrm{conv}(R_f(W))$

from $\mathbf{0}$ even though $\mathbf{0}$ is in the closure of $\mathrm{conv}(R_f(W))$. This implies that $\mathcal{H}$ is not a closed hyperplane of $\mathcal{W}$, and in particular the function $s \mapsto \sum_{r \in W} \psi(r) s_r$ is not continuous with respect to the Euclidean norm on $\mathcal{W}$.

A nice answer to our question is given by considering a different norm on $\mathcal{W}$. We endow $\mathcal{W}$ with the norm $\|\cdot\|_H$ defined by

$$\|s\|_H = |s_0| + \sum_{r \in W \setminus \{0\}} \|r\| |s_r|.$$

It is straightforward to show that $\|\cdot\|_H$ is indeed a norm. Given $A \subset \mathcal{W}$, we denote by $\bar{A}$ the closure of A with respect to the norm $\|\cdot\|_H$.

Let $\mathcal{B}_W$ be the family of all maximal lattice-free convex sets of W with f in their interior.

Theorem 27.

$$\overline{\mathrm{conv}}(R_f(W)) = \left\{ s \in \mathcal{V} \mid \begin{array}{ll} \sum_{r \in W} \psi_B(r) s_r \geq 1 & B \in \mathcal{B}_W \\ s_r \geq 0 & r \in W \end{array} \right\}.$$

Note that Theorems 26 and 27 are new even when $W = \mathbb{R}^q$. Even though data of integer programs are typically rational and studying the infinite relaxation (3.1) for $W = \mathbb{Q}^q$ seems natural, some of its extreme inequalities arise from maximal lattice free convex sets with irrational facets [34]. Therefore the more natural setting for (3.1) is in fact $W = \mathbb{R}^q$.

The rest of this section is organized as follows. In Section 3.1.1 we will state and prove the natural extensions of Theorems 24 and 25 for general lattices. In Section 3.1.2 we prove Theorem 26, while in Section 3.1.3 we prove Theorem 27.

3.1.1 Maximal Lattice-free Convex Sets

Given $y \in \mathbb{R}^n$ and $\varepsilon > 0$, we will denote by $B_\varepsilon(y)$ the open ball centered at y of radius ε. Given an affine space W of $\mathbb{R}^n$ and a set $S \subseteq W$, we denote by $\mathbf{int}_W(S)$ the interior of S with respect to the topology induced on W by $\mathbb{R}^n$, namely $\mathbf{int}_W(S)$ is the set of points $x \in S$ such that $B_\varepsilon(x) \cap W \subset S$ for some $\epsilon > 0$. We denote by $\mathbf{relint}(S)$ the relative interior of S, that is $\mathbf{relint}(S) = \mathbf{int}_{\mathrm{aff}(S)}(S)$.

Definition 28. *Let Λ be a lattice of a linear space V of $\mathbb{R}^n$, and let W be a linear space of $\mathbb{R}^n$ containing V. A set $S \subset \mathbb{R}^n$ is said to be a* Λ-free convex set of W *if $S \subset W$, S*

is convex and $\Lambda \cap \mathbf{int}_W(S) = \emptyset$*, and* S *is said to be a* maximal Λ-free convex set of W *if it is not properly contained in any* Λ*-free convex set.*

The next two theorems are restatements of Theorems 24 and 25 for general lattices.

Theorem 29. *Let* Λ *be a lattice of a linear space* V *of* $\mathbb{R}^n$*, and let* W *be a linear space of* $\mathbb{R}^n$ *containing* V*. A set* $S \subset \mathbb{R}^n$ *is a maximal* Λ*-free convex set of* W *if and only if one of the following holds:*

(i) S *is a polyhedron in* W*,* $\dim(S) = \dim(W)$*,* $S \cap V$ *is a maximal* Λ*-free convex set of* V*, and for every facet* F *of* S*,* $F \cap V$ *is a facet of* $S \cap V$*;*

(ii) S *is an affine hyperplane of* W *of the form* $S = v + L$ *where* $v \in S$ *and* $L \cap V$ *is a hyperplane of* V *that is not a lattice subspace of* V*;*

(iii) S *is a half-space of* W *that contains* V *on its boundary.*

Theorem 30. *Let* Λ *be a lattice of a linear space* V *of* $\mathbb{R}^n$*. A set* $S \subset \mathbb{R}^n$ *is a maximal* Λ*-free convex set of* V *if and only if one of the following holds:*

(i) S *is a polyhedron of the form* $S = P + L$ *where* P *is a polytope,* L *is a lattice-subspace of* V*,* $\dim(S) = \dim(P) + \dim(L) = \dim(V)$*,* S *does not contain any point of* Λ *in its interior and there is a point of* Λ *in the relative interior of each facet of* S*;*

(ii) $\dim(S) < \dim(V)$*,* S *is an affine hyperplane of* V *of the form* $S = v + L$ *where* $v \in S$ *and* L *is not a lattice-subspace of* V*.*

Proof of Theorem 29

We assume Theorem 30 holds. Its proof will be given in the next section.

($\Rightarrow$) Let S be a maximal Λ-free convex set of W. We show that one of $(i)-(iii)$ holds. If $V = W$, then (iii) cannot occur and either (i) or (ii) follows from Theorem 30. Thus we assume $V \subset W$.

Assume first that $\dim(S) < \dim(W)$. Then there exists a hyperplane H of W containing S, and since $\mathbf{int}_W(H) = \emptyset$, then $S = H$ by maximality of S. Since S is a

hyperplane of W, then either $V \subseteq S$ or $S \cap V$ is a hyperplane of V. If $V \subseteq S$, then let K be one of the two half spaces of W separated by S. Then $\mathbf{int}_W(K) \cap \Lambda = \emptyset$, contradicting the maximality of S. Hence $S \cap V$ is a hyperplane of V. We show that $P = S \cap V$ is a maximal Λ-free convex set of V. Indeed, let K be a convex set in V such that $\mathbf{int}_V(K) \cap \Lambda = \emptyset$ and $P \subseteq K$. Since $\mathrm{conv}(S \cup K) \cap V = K$, then $\mathbf{int}_W(\mathrm{conv}(S \cup K) \cap \Lambda) = \emptyset$. By maximality of S, $S = \mathrm{conv}(S \cup K)$, hence $P = K$. Given $v \in P$, $S = v + L$ for some hyperplane L of W, and $P = v + (L \cap V)$. Applying Theorem 30 to P, we get that $L \cap V$ is not a lattice subspace of V, and case (ii) holds.

So we may assume $\dim(S) = \dim(W)$. Since S is convex, then $\mathbf{int}_W(S) \neq \emptyset$. We consider two cases.

Case 1. $\mathbf{int}_W(S) \cap V = \emptyset$.

Since $\mathbf{int}_W(S)$ and V are nonempty disjoint convex sets, there exists a hyperplane separating them, i.e. there exist $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ such that $\alpha x \geq \beta$ for every $x \in S$ and $\alpha x \leq \beta$ for every $x \in V$. Since V is a linear space, then $\alpha x = 0$ for every $x \in V$, hence $\beta \geq 0$. Then the half space $H = \{x \in W \mid \alpha x \geq 0\}$ contains S and V lies on the boundary of H. Hence H is a maximal Λ-free convex set of W containing S, therefore $S = H$ by the maximality assumption, so (iii) holds.

Case 2. $\mathbf{int}_W(S) \cap V \neq \emptyset$.

We claim that

$$\mathbf{int}_W(S) \cap V = \mathbf{int}_V(S \cap V). \tag{3.4}$$

To prove this claim, notice that the direction $\mathbf{int}_W(S) \cap V \subseteq \mathbf{int}_V(S \cap V)$ is straightforward. Conversely, let $x \in \mathbf{int}_V(S \cap V)$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \cap V \subseteq S$. Since $\mathbf{int}_W(S) \cap V \neq \emptyset$, there exists $y \in \mathbf{int}_W(S) \cap V$. Then there exists ε' such that $B_{\varepsilon'}(y) \cap W \subseteq S$. Since S is convex, the set $K = \mathrm{conv}((B_\varepsilon(x) \cap V) \cup (B_{\varepsilon'}(y) \cap W))$ is contained in S. Clearly $x \in \mathbf{int}_W(K)$, thus $x \in \mathbf{int}_W(S) \cap V$.

Let $P = S \cap V$. By (3.4) and because $\mathbf{int}_W(S) \cap \Lambda = \emptyset$, we have $\mathbf{int}_V(P) \cap \Lambda = \emptyset$. We show that P is a maximal Λ-free convex set of V. Indeed, let K be a convex set in V such that $\mathbf{int}_V(K) \cap \Lambda = \emptyset$ and $P \subseteq K$. Since $\mathrm{conv}(S \cup K) \cap V = K$, Claim (3.4) implies that $\mathbf{int}_W(\mathrm{conv}(S \cup K)) \cap \Lambda = \emptyset$. By maximality, $S = \mathrm{conv}(S \cup K)$, hence $P = K$.

Since $\dim(P) = \dim(V)$, by Theorem 30 applied to P, P is a polyhedron with a point of Λ in the relative interior of each of its facets. Let $F_1, \ldots, F_t$ be the facets of P. For $i = 1, \ldots, t$, let z_i be a point in $\mathbf{relint}(F_i) \cap \Lambda$. By (3.4), $z_i \notin \mathbf{int}_W(S)$. By

the separation theorem, there exists a half-space H_i of W containing $\mathbf{int}_W(S)$ such that $z_i \notin \mathbf{int}_W(H_i)$. Notice that F_i is on the boundary of H_i. Then $S \subseteq \cap_{i=1}^t H_i$. By construction $\mathbf{int}_W(\cap_{i=1}^t H_i) \cap \Lambda = \emptyset$, hence by maximality of S, $S = \cap_{i=1}^t H_i$. For every $j = 1, \ldots, t$, $\mathbf{int}_W(\cap_{i \neq j} H_i)$ contains z_j. Therefore H_j defines a facet of S for $j = 1, \ldots, t$.

($\Leftarrow$) Let S be a set in $\mathbb{R}^n$ satisfying one of $(i), (ii), (iii)$. Clearly S is a convex set in W and $\mathbf{int}_W(S) \cap \Lambda = \emptyset$, so we only need to prove maximality. If S satisfies (iii), then this is immediate. If S satisfies (i) or (ii), suppose that there exists a closed convex set $K \subset W$ strictly containing S such that $\mathbf{int}_W(K) \cap \Lambda = \emptyset$. Let $w \in K \setminus S$. Then $\mathrm{conv}(S \cup \{w\}) \subseteq K$. We claim that the inclusion $S \cap V \subset \mathrm{conv}(S \cup \{w\}) \cap V$ is strict. This is clear when S is a hyperplane satisfying (ii). When S is a polyhedron satisfying (i), the claim follows from the fact that each facet F of S has the property that $F \cap V$ is a facet of $S \cap V$. Thus $S \cap V \subset \mathrm{conv}(S \cup \{w\}) \cap V$. By maximality of $S \cap V$, the set $\mathbf{int}_V(\mathrm{conv}(S \cup \{w\}) \cap V)$ contains a point in Λ. Now $\mathrm{conv}(S \cup \{w\}) \subseteq K$ implies that $\mathbf{int}_W(K)$ contains a point of Λ, a contradiction.

Proof of Theorem 30

Throughout this section, Λ is a lattice of a linear space V. To simplify notation, given $S \subseteq \mathbb{R}^n$, we denote $\mathbf{int}_V(S)$ simply by $\mathbf{int}(S)$.

The following lemma proves the "only if" part of Theorem 30 when S is bounded and full-dimensional.

Lemma 31. *Let Λ be a lattice of a linear space V of $\mathbb{R}^n$. Let $S \subset V$ be a bounded maximal Λ-free convex set with $\dim(S) = \dim(V)$. Then S is a polytope with a point of Λ in the relative interior of each of its facets.*

Proof. Since S is bounded, there exist integers L, U such that S is contained in the box $B = \{x \in \mathbb{R}^d \mid L \leq x_i \leq U\}$. For each $y \in \Lambda \cap B$, since S is convex there exists a closed half-space H^y of V such that $S \subseteq H^y$ and $y \notin \mathbf{int}(H^y)$. By Theorem 3, $B \cap \Lambda$ is finite, therefore $\bigcap_{y \in B \cap \Lambda} H^y$ is a polyhedron. Thus $P = \bigcap_{y \in B \cap \Lambda} H^y \cap B$ is a polytope and by construction $\Lambda \cap \mathbf{int}(P) = \emptyset$. Since $S \subseteq B$ and $S \subseteq H^y$ for every $y \in B \cap \Lambda$, it follows that $S \subseteq P$. By maximality of S, $S = P$, therefore S is a polytope. We only need to show that S has a point of Λ in the relative interior of each of its facets. Let $F_1, \ldots, F_t$ be the facets of S, and let $H_i = \{x \in V \mid \alpha_i x \leq \beta_i\}$ be the closed half-space defining F_i, $i = 1, \ldots, t$. Then $S = \bigcap_{i=1}^t H_i$. Suppose, by contradiction, that one of the facets of S, say F_t, does not contain a point of Λ

in its relative interior. Given $\varepsilon > 0$, the polyhedron $S' = \{x \in V \mid \alpha_i x \leq \beta_i,\ i = 1, \ldots, t-1,\ \alpha_t x \leq \beta_t + \varepsilon\}$ contains points of Λ in its interior by the maximality of S. By Theorem 3, $\mathbf{int}(S')$ has a finite number of points in Λ, hence there exists one minimizing $\alpha_t x$, say z. By construction, the polytope $S' = \{x \in V \mid \alpha_i x \leq \beta_i,\ i = 1, \ldots, t-1,\ \alpha_t x \leq \alpha_t z\}$ does not contain any point of Λ in its interior and properly contains S, contradicting the maximality of S. $\square$

Lemma 32. *Let L be a linear subspace of V with $\dim(L) = \dim(V) - 1$, and let $v \in V$. Then $v + L$ is a maximal Λ-free convex set if and only if L is not a lattice subspace of V.*

Proof. ($\Rightarrow$) Let $S = v + L$ and assume that S is a maximal Λ-free convex set. Suppose by contradiction that L is a lattice-subspace. Then there exists a basis $a_1, \ldots, a_m$ of Λ such that $a_1, \ldots, a_{m-1}$ is a basis of L. Thus $S = \{\sum_{i=1}^m x_i a_i \mid x_m = \beta\}$ for some $\beta \in \mathbb{R}$. Then, $K = \{\sum_{i=1}^m x_i a_i \mid \lceil \beta - 1 \rceil \leq x_m \leq \lceil \beta \rceil\}$ strictly contains S and $\mathbf{int}(K) \cap \Lambda = \emptyset$, contradicting the maximality of S.

($\Leftarrow$) Assume L is not a lattice-subspace of V. Since $S = v + L$ is an affine hyperplane of V, $\mathbf{int}(S) = \emptyset$, thus $\mathbf{int}(S) \cap \Lambda = \emptyset$, hence we only need to prove that S is maximal with such property. Suppose not, and let K be a maximal convex set in V such that $\mathbf{int}(K) \cap \Lambda = \emptyset$ and $S \subset K$. Then by maximality K is closed. Let $w \in K \setminus S$. Since K is convex and closed, then $K \supseteq \mathrm{conv}(\{v, w\}) + L$. Let ε be the distance between $v + L$ and $w + L$, and δ be the distance of $\mathrm{conv}(\{v, w\}) + L$ from the origin. By Lemma 8, since L is not a lattice-subspace of V, there exists a vector $y \in \Lambda \setminus L$ at distance $\bar{\varepsilon} < \varepsilon$ from L. Let $z = (\lfloor \frac{\delta}{\bar{\varepsilon}} \rfloor + 1)y$. By definition, z is strictly between $v + L$ and $w + L$, hence $z \in \mathbf{int}(K)$. Since z is an integer multiple of $y \in \Lambda$, then $z \in \Lambda$, a contradiction. $\square$

We are now ready to prove Lovász's Theorem.

Proof of Theorem 30. ($\Leftarrow$) If S satisfies (ii), then by Lemma 32, S is a maximal Λ-free convex set. If S satisfies (i), then, since $\mathbf{int}(S) \cap \Lambda = \emptyset$, we only need to show that S is maximal. Suppose not, and let K be a convex set in V such that $\mathbf{int}(K) \cap \Lambda = \emptyset$ and $S \subset K$. Given $y \in K \setminus S$, there exists a hyperplane H separating y from S such that $F = S \cap H$ is a facet of S. Since K is convex and $S \subset K$, then $\mathrm{conv}(S \cup \{y\}) \subseteq K$. Since $\dim(S) = \dim(V)$, $F \subset S$ hence the $\mathbf{relint}(F) \subset \mathbf{int}(K)$. By assumption, there exists $x \in \Lambda \cap \mathbf{relint}(F)$, so $x \in \mathbf{int}(K)$, a contradiction.

($\Rightarrow$) Let S be a maximal Λ-free convex set. We show that S satisfies either (i) or (ii). Observe that, by maximality, S must be closed.

If $\dim(S) < \dim(V)$, then S is contained in some affine hyperplane H. Since $\mathbf{int}(H) = \emptyset$, we have $S = H$ by maximality of S, therefore $S = v + L$ where $v \in S$ and L is a hyperplane in V. By Lemma 32, (ii) holds.

Therefore we may assume that $\dim(S) = \dim(V)$. In particular, since S is convex, $\mathbf{int}(S) \neq \emptyset$. By Lemma 31, if S is bounded, (i) holds. Hence we may assume that S is unbounded. Let C be the recession cone of S and L the lineality space of S. By Proposition 11, $C \neq 0$.

Claim 1. $L = C$.

Let $r \in C$, $r \neq 0$. We only need to show that $S + \langle r \rangle$ is Λ-free; by maximality of S this will imply that $S = S + \langle r \rangle$. Suppose there exists $y \in \mathbf{int}(S + \langle r \rangle) \cap \Lambda$. We show that $y \in \mathbf{int}(S) + \langle r \rangle$. Suppose not. Then $(y + \langle r \rangle) \cap \mathbf{int}(S) = \emptyset$, which implies that there is a hyperplane H separating the line $y + \langle r \rangle$ and $S + \langle r \rangle$. This contradicts $y \in \mathbf{int}(S + \langle r \rangle)$. This shows $y \in \mathbf{int}(S) + \langle r \rangle$. Thus there exists $\bar{\lambda}$ such that $\bar{y} = y + \bar{\lambda} r \in \mathbf{int}(S)$, i.e. there exists $\varepsilon > 0$ such that $B_\varepsilon(\bar{y}) \cap V \subset S$. Since $y \in \Lambda$, then $y \notin \mathbf{int}(S)$, and thus, since $\bar{y} \in \mathbf{int}(S)$ and $r \in C$, we must have $\bar{\lambda} > 0$. Since $r \in C$, then $B_\varepsilon(\bar{y}) + \{\lambda r \mid \lambda \geq 0\} \subset S$. Since $y \in \Lambda$, by Lemma 7 there exists $z \in \Lambda$ at distance less than ε from the half line $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$. Thus $z \in B_\varepsilon(\bar{y}) + \{\lambda r \mid \lambda \geq 0\}$, hence $z \in \mathbf{int}(S)$, a contradiction. $\diamond$

Let $P = \mathrm{proj}_{L^\perp}(S)$ and $\Lambda' = \mathrm{proj}_{L^\perp}(\Lambda)$. By Claim 1, $S = P + L$ and $P \subset L^\perp \cap V$ is a bounded set. Furthermore, $\dim(S) = \dim(P) + \dim(L) = \dim(V)$ and $\dim(P) = \dim(L^\perp \cap V)$. Notice that $\mathbf{int}(S) = \mathbf{relint}(P) + L$, hence $\mathbf{relint}(P) \cap \Lambda' = \emptyset$. Furthermore P is inclusionwise maximal among the convex sets of $L^\perp \cap V$ without points of Λ' in the relative interior: if not, given a convex set $K \subseteq L^\perp \cap V$ strictly containing P and with no point of Λ' in its relative interior, we have $S = P + L \subset K + L$, and $K + L$ does not contain any point of Λ in its interior, contradicting the maximality of S.

Claim 2. *L is a lattice-subspace of V.*

By contradiction, suppose L is not a lattice-subspace of V. Then, by Lemma 8, for every $\varepsilon > 0$ there exists $y \in \Lambda' \setminus \{0\}$ such that $\|y\| < \varepsilon$. Let V_ε be the linear subspace of $L^\perp \cap V$ generated by the points in $\{y \in \Lambda' \mid \|y\| < \varepsilon\}$. Then $\dim(V_\varepsilon) > 0$.

Notice that, given $\varepsilon' > \varepsilon'' > 0$, then $V_{\varepsilon'} \supseteq V_{\varepsilon''} \supset \{0\}$, hence there exists $\varepsilon_0 > 0$ such that $V_\varepsilon = V_{\varepsilon_0}$ for every $\varepsilon < \varepsilon_0$. Let $U = V_{\varepsilon_0}$.

By definition, Λ' is dense in U (i.e. for every $\varepsilon > 0$ and every $x \in U$ there exists $y \in \Lambda'$ such that $\|x - y\| < \varepsilon$). Thus, since $\mathbf{relint}(P) \cap \Lambda' = \emptyset$, we also have $\mathbf{relint}(P) \cap U = \emptyset$. Since $\dim(P) = \dim(L^\perp \cap V)$, it follows that $\mathbf{relint}(P) \cap (L^\perp \cap V) \neq \emptyset$, so in particular U is a proper subspace of $L^\perp \cap V$.

Let $Q = \mathrm{proj}_{(L+U)^\perp}(P)$ and $\Lambda'' = \mathrm{proj}_{(L+U)^\perp}(\Lambda')$. We show that $\mathbf{relint}(Q) \cap \Lambda'' = \emptyset$. Suppose not, and let $y \in \mathbf{relint}(Q) \cap \Lambda''$. Then, $y + w \in \Lambda'$ for some $w \in U$. Furthermore, we claim that $y + w' \in \mathbf{relint}(P)$ for some $w' \in U$. Indeed, suppose no such w' exists. Then $(y + U) \cap (\mathbf{relint}(P) + U) = \emptyset$. So there exists a hyperplane H in $L^\perp \cap V$ separating $y + U$ and $P + U$. Therefore the projection of H onto $(L+U)^\perp$ separates y and Q, contradicting $y \in \mathbf{relint}(Q)$. Thus $z = y + w' \in \mathbf{relint}(P)$ for some $w' \in U$. Since $z \in \mathbf{relint}(P)$, there exists $\bar{\varepsilon} > 0$ such that $B_{\bar{\varepsilon}}(z) \cap (L^\perp \cap V) \subset \mathbf{relint}(P)$. Since Λ' is dense in U and $y + w \in \Lambda'$, it follows that Λ' is dense in $y + U$. Hence, since $z \in y + U$, there exists $\bar{x} \in \Lambda'$ such that $\|\bar{x} - z\| < \bar{\varepsilon}$, hence $\bar{x} \in \mathbf{relint}(P)$, a contradiction. This shows $\mathbf{relint}(Q) \cap \Lambda'' = \emptyset$.

Finally, since $\mathbf{relint}(Q) \cap \Lambda'' = \emptyset$, then $\mathbf{int}(Q + L + U) \cap \Lambda = \emptyset$. Furthermore $P \subseteq Q + U$, therefore $S \subseteq Q + L + U$. By the maximality of S, $S = Q + L + U$ hence the lineality space of S contains $L + U$, contradicting the fact that L is the lineality space of S and $U \neq \{0\}$. $\diamond$

Since L is a lattice-subspace of V, Λ' is a lattice of $L^\perp \cap V$ by Lemma 5. Since P is a bounded maximal Λ'-free convex set, it follows from Lemma 31 that P is a polytope with a point of Λ' in the relative interior of each of its facets, therefore $S = P + L$ has a point of Λ in the relative interior of each of its facets, and (i) holds. $\square$

Zorn's Lemma implies the following simple fact.

Proposition 33. *Every Λ-free convex set of V is contained in some maximal Λ-free convex set of V.*

3.1.2 Minimal Valid Inequalities

In this section we will prove Theorem 26. For ease of notation, we denote $R_f(W)$ simply by R_f in this section. It is easy to see that a linear functional $\Psi : \mathcal{W} \to \mathbb{R}$ is

of the form

$$\Psi(s) = \sum_{r \in W} \psi(r) s_r, \quad s \in \mathcal{W} \tag{3.5}$$

for some $\psi : W \to \mathbb{R}$. Throughout the rest of this chapter, capitalized Greek letters indicate linear functions from $\mathcal{W}$ to $\mathbb{R}$, while the corresponding lowercase letters indicate functions from W to $\mathbb{R}$ as defined in (3.5).

Lemma 34. *Let $\Psi(s) \geq \alpha$ be a valid linear inequality for R_f. Then $\Psi(s) \geq \alpha$ is dominated by a valid linear inequality $\Psi'(s) \geq \alpha$ for R_f such that ψ' is sublinear.*

Proof: We first prove the following.

Claim 1. *For every $s \in \mathcal{W}$ such that $\sum_{r \in W} r s_r = 0$ and $s_r \geq 0$, $r \in W$, we have $\sum_{r \in W} \psi(r) s_r \geq 0$.*

Suppose not. Then there exists $s \in \mathcal{W}$ such that $\sum_{r \in W} r s_r = 0$, $s_r \geq 0$ for all $r \in W$ and $\sum_{r \in W} \psi(r) s_r < 0$. Let $\bar{x}$ be an integral point in W. For any $\lambda > 0$, we define $s^\lambda \in \mathcal{W}$ by

$$s_r^\lambda = \begin{cases} 1 + \lambda s_r & \text{for } r = \bar{x} - f \\ \lambda s_r & \text{otherwise.} \end{cases}$$

Since $f + \sum_{r \in W} r s_r^\lambda = \bar{x}$, it follows that s^λ is in R_f. Furthermore $\sum_{r \in W} \psi(r) s_r^\lambda = \psi(\bar{x} - f) + \lambda(\sum_{r \in W} \psi(r) s_r)$. Therefore $\sum_{r \in W} \psi(r) s_r^\lambda$ goes to $-\infty$ as λ goes to $+\infty$. $\diamond$

We define, for all $\bar{r} \in W$,

$$\psi'(\bar{r}) = \inf\{\sum_{r \in W} \psi(r) s_r \,|\, \bar{r} = \sum_{r \in W} r s_r,\ s \in \mathcal{W},\ s_r \geq 0 \text{ for all } r \in W\}.$$

By Claim 1, $\sum_{r \in W} \psi(r) s_r \geq -\psi(-\bar{r})$ for all $s \in \mathcal{W}$ such that $\bar{r} = \sum_{r \in W} r s_r$ and $s_r \geq 0$ for all $r \in W$. Thus the infimum in the above equation is finite and the function ψ' is well defined. Note also that $\psi'(\bar{r}) \leq \psi(\bar{r})$ for all $\bar{r} \in W$, as follows by considering $s \in \mathcal{W}$ defined by $s_{\bar{r}} = 1$, $s_r = 0$ for all $r \in W$, $r \neq \bar{r}$.

Claim 2. *The function ψ' is sublinear*

Note first that $\psi'(0) = 0$. Indeed, Claim 1 implies $\psi'(0) \geq 0$, while choosing $s_r = 0$ for all $r \in W$ shows $\psi'(0) \leq 0$.

Next we show that ψ' is positively homogeneous. To prove this, let $\bar{r} \in W$ and $s \in \mathcal{W}$ such that $\bar{r} = \sum_{r \in W} r s_r$ and $s_r \geq 0$ for all $r \in W$. Let $\gamma = \sum_{r \in W} \psi(r) s_r$. For every $\lambda > 0$, $\lambda \bar{r} = \sum_{r \in W} r(\lambda s_r)$, $\lambda s_r \geq 0$ for all $r \in W$, and $\sum_{r \in W} \psi(r)(\lambda s_r) = \lambda \gamma$. Therefore $\psi'(\lambda \bar{r}) = \lambda \psi'(r)$.

Finally, we show that ψ' is convex. Suppose by contradiction that there exist $r', r'' \in W$ and $0 < \lambda < 1$ such that $\psi'(\lambda r' + (1-\lambda) r'') > \lambda \psi'(r') + (1-\lambda)\psi'(r'') + \epsilon$ for some positive ϵ. By definition of ψ', there exist $s', s'' \in \mathcal{W}$ such that $r' = \sum_{r \in W} r s'_r$, $r'' = \sum_{r \in W} r s''_r$, $s'_r, s''_r \geq 0$ for all $r \in W$, $\sum_{r \in W} \psi(r) s'_r < \psi'(r') + \epsilon$ and $\sum_{r \in W} \psi(r) s''_r < \psi'(r'') + \epsilon$. Since $\sum_{r \in W} r(\lambda s'_r + (1-\lambda) s''_r) = \lambda r' + (1-\lambda) r''$, it follows that $\psi'(\lambda r' + (1-\lambda) r'') \leq \sum_{r \in W} \psi(r)(\lambda s'_r + (1-\lambda) s''_r) < \lambda \psi'(r') + (1-\lambda)\psi'(r'') + \epsilon$, a contradiction. $\diamond$

Claim 3. *The inequality $\sum_{r \in W} \psi'(r) s_r \geq \alpha$ is valid for R_f.*

Suppose there exists $\bar{s} \in R_f$ such that $\sum_{r \in W} \psi'(r) \bar{s}_r \leq \alpha - \epsilon$ for some positive ϵ. Let $\{r^1, \ldots, r^k\} = \{r \in W \mid \bar{s}_r > 0\}$. For every $i = 1, \ldots, k$, there exists $s^i \in W$ such that $r^i = \sum_{r \in W} r s^i_r$, $s^i_r \geq 0$, $r \in W$, and $\sum_{r \in W} \psi(r) s^i_r < \psi'(r^i) + \epsilon/(k \bar{s}_{r^i})$.

Let $\tilde{s} = \sum_{i=1}^k \bar{s}_{r^i} s^i$. Then

$$\sum_{r \in W} r \tilde{s}_r = \sum_{r \in W} \sum_{i=1}^k r \bar{s}_{r^i} s^i_r = \sum_{i=1}^k \bar{s}_{r^i} \sum_{r \in W} r s^i_r = \sum_{i=1}^k r^i \bar{s}_{r^i} = \sum_{r \in W} r \bar{s}_r,$$

hence $\tilde{s} \in R_f$. Therefore $\sum_{r \in W} \psi(r) \tilde{s}_r \geq \alpha$ since $\sum_{r \in W} \psi(r) s_r \geq \alpha$ is valid for R_f. Now

$$\begin{aligned} \sum_{r \in W} \psi(r) \tilde{s}_r &= \sum_{r \in W} \sum_{i=1}^k \psi(r) \bar{s}_{r^i} s^i_r = \sum_{i=1}^k \bar{s}_{r^i} \sum_{r \in W} \psi(r) s^i_r \\ &< \sum_{i=1}^k \bar{s}_{r^i} (\psi'(r^i) + \epsilon/(k \bar{s}_{r^i})) = \sum_{r \in W} \psi'(r^i) \bar{s}_{r^i} + \epsilon \leq \alpha, \end{aligned}$$

a contradiction. $\square$

Lemma 35. *Let $\Psi(s) \geq \alpha$ and $\Psi'(s) \geq \alpha'$ be two equivalent valid linear inequalities for R_f. (i) The function ψ is sublinear if and only if ψ' is sublinear.*
(ii) Inequality $\Psi(s) \geq \alpha$ is dominated by a minimal valid linear inequality if and only if $\Psi'(s) \geq \alpha'$ is dominated by a minimal valid linear inequality. In particular, $\Psi(s) \geq \alpha$ is minimal if and only if $\Psi'(s) \geq \alpha'$ is minimal.

Proof. Since $\Psi(s) \geq \alpha$ and $\Psi'(s) \geq \alpha'$ are equivalent, by definition there exist $\rho > 0$ and $\lambda \in \mathbb{R}^\ell$, such that $\psi(r) = \rho\psi'(r) + \lambda^T Cr$ and $\alpha = \rho\alpha' + \lambda^T(d - Cf)$. This proves (i).

Point (ii) follows from the fact that, given a function $\bar{\psi}'$ such that $\bar{\psi}'(r) \leq \psi'(r)$ for every $r \in W$, then the function $\bar{\psi}$ defined by $\bar{\psi}(r) = \rho\bar{\psi}'(r) + \lambda^T Cr$, $r \in W$, satisfies $\bar{\psi}(r) \leq \psi(r)$ for every $r \in W$. Furthermore $\bar{\psi}(r) < \psi(r)$ if and only if $\bar{\psi}'(r) < \psi'(r)$. □

Given a nontrivial valid linear inequality $\Psi(s) \geq \alpha$ for R_f such that ψ is sublinear, we consider the set

$$B_\psi = \{x \in f + W \mid \psi(x - f) \leq \alpha\}.$$

Since ψ is continuous (Proposition 14), B_ψ is closed. Since ψ is convex, B_ψ is convex. Since ψ defines a valid inequality, B_ψ is lattice-free. Indeed the interior of B_ψ is $\mathbf{int}(B_\psi) = \{x \in f+W \,:\, \psi(x-f) < \alpha\}$. Its boundary is $\mathbf{bd}(B_\psi) = \{x \in f+W \,:\, \psi(x - f) = \alpha\}$, and its recession cone is $\mathrm{rec}(B_\psi) = \{x \in f + W \,:\, \psi(x - f) \leq 0\}$. Note that f is in the interior of B_ψ if and only if $\alpha > 0$ and f is on the boundary if and only if $\alpha = 0$.

Remark 36. *Given a linear inequality of the form $\Psi(s) \geq 1$ such that $\psi(r) \geq 0$ for all $r \in W$,*

$$\psi(r) = \inf\{t > 0 \mid f + t^{-1}r \in B_\psi\}, \quad r \in W.$$

Proof. Let $r \in W$. If $\psi(r) > 0$, let t be the minimum positive number such that $f + t^{-1}r \in B_\psi$. Then $f + t^{-1}r \in \mathbf{bd}(B_\psi)$, hence $\psi(t^{-1}r) = 1$ and by positive homogeneity $\psi(r) = t$. If $\psi(r) = 0$, then $r \in \mathrm{rec}(B_\psi)$, hence $f + t^{-1}r \in B_\psi$ for every $t > 0$, thus the infimum in the above equation is 0. □

This remark shows that, if ψ is nonnegative, then it is the gauge of the convex set $B_\psi - f$.

Before proving Theorem 26, we recall the definition of ρ_K for a given closed convex set K with the origin in its interior from Section 2.2.1.

$$\rho_K(r) = \sup_{y \in \hat{K}} ry, \quad \text{for all } r \in W. \tag{3.6}$$

We recall that ρ_K is sublinear.

Remark 37. *Let $K \subset W$ be a polyhedron containing the origin in its interior. Let $a_1, \ldots, a_t \in W$ such that $K = \{r \in W \mid a_i r \leq 1, i = 1, \ldots, t\}$. Then $\rho_K(r) = \max_{i=1,\ldots,t} a_i r$.*

Proof. The polar of K is $K^* = \mathrm{conv}\{0, a_1, \ldots, a_t\}$ (see Theorem 9.1 in Schrijver [58]). Furthermore, $\hat{K}$ is the union of all the facets of K^* that do not contain the origin, therefore

$$\rho_K(r) = \sup_{y \in \hat{K}} yr = \max_{i=1,\ldots,t} a_i r$$

for all $r \in W$. □

Remark 38. *Let B be a closed lattice-free convex set in $f + W$ with f in its interior, and let $K = B - f$. Then the inequality $\sum_{r \in W} \rho_K(r) s_r \geq 1$ is valid for R_f.*

Proof: Let $s \in R_f$. Then $x = f + \sum_{r \in W} r s_r$ is integral, therefore $x \notin \mathbf{int}(B)$ because B is lattice-free. By Theorem 15, $\rho_K(x - f) \geq 1$. Thus

$$1 \leq \rho_K(\sum_{r \in W} r s_r) \leq \sum_{r \in W} \rho_K(r s_r) \leq \sum_{r \in W} \rho_K(r) s_r,$$

where the second inequality follows from the subadditivity of ρ_K and the last from the positive homogeneity. □

Lemma 39. *Given a maximal lattice-free convex set B of $f + W$ containing f in its interior, $\Psi_B(s) \geq 1$ is a minimal valid inequality for R_f.*

Proof. Let $\Psi(s) \geq 1$ be a valid linear inequality for R_f such that $\psi(r) \leq \psi_B(r)$ for all $r \in W$. Then $B_\psi \supset B$ and B_ψ is lattice-free. By maximality of B, $B = B_\psi$. By Theorem 15 and Remark 37, $\psi_B(r) \leq \psi(r)$ for all $r \in W$, proving $\psi = \psi_B$. □

Proof of Theorem 26.

Let $\Psi(s) \geq \alpha$ be a nontrivial valid linear inequality for R_f. By Lemma 34, we may assume that ψ is sublinear.

Claim 1. *If $\mathbf{int}(B_\psi) \cap V = \emptyset$, then $\Psi(s) \geq \alpha$ is trivial.*

Suppose $\mathbf{int}(B_\psi) \cap V = \emptyset$ and let $s \in \mathcal{V}$ such that $s_r \geq 0$ for every $r \in W$. Let $x = f + \sum_{r \in W} r s_r$. Since $s \in \mathcal{V}$, $x \in V$, so $x \notin \mathbf{int}(B_\psi)$. This implies

$$\alpha \leq \psi(x - f) = \psi(\sum_{r \in W} r s_r) \leq \sum_{r \in W} \psi(r) s_r = \Psi(s),$$

where the last inequality follows from the sublinearity of ψ. $\diamond$

Claim 2. *If $f \in V$ and $\alpha \leq 0$, then $\mathbf{int}(B_\psi) \cap V = \emptyset$.*

Suppose $f \in V$, $\alpha \leq 0$ but $\mathbf{int}(B_\psi) \cap V \neq \emptyset$. Then $\dim(\mathbf{int}(B_\psi) \cap V) = \dim(V)$, hence $\mathbf{int}(B_\psi) \cap V$ contains a set X of $\dim(V) + 1$ affinely independent points. For every $x \in X$ and every $\lambda > 0$, $\psi(\lambda(x - f)) = \lambda \psi(x - f) < 0$, where the last inequality is because $x \in \mathbf{int}(B_\psi)$. Hence the set $\Gamma = f + \mathrm{cone}\{x - f \mid x \in X\}$ is contained in $\mathbf{int}(B_\psi)$. Since Γ has dimension equal to $\dim(V)$ and V is the convex hull of its integral points, $\Gamma \cap \mathbb{Z}^q \neq 0$, contradicting the fact that B_ψ has no integral point in its interior. $\diamond$

Claim 3. *If $f \notin V$, then there exists a valid linear inequality $\Psi'(s) \geq 1$ for R_f equivalent to $\Psi(s) \geq \alpha$.*

Since $f \notin V$, $Cf \neq d$, hence there exists a row c_i of C such that $d_i - c_i f \neq 0$. Let $\lambda = (1 - \alpha)(d_i - c_i f)^{-1}$, and define $\psi'(r) = \psi(r) + \lambda c_i r$ for every $r \in W$. The inequality $\Psi'(s) \geq 1$ is equivalent to $\Psi(s) \geq \alpha$. $\diamond$

Thus, by Claims 1, 2 and 3 there exists a valid linear inequality $\Psi'(s) \geq 1$ for R_f equivalent to $\Psi(s) \geq \alpha$. By Lemma 35, ψ' is sublinear and $\Psi(s) \geq \alpha$ is dominated by a minimal valid linear inequality if and only if $\Psi'(s) \geq \alpha'$ is dominated by a minimal valid linear inequality. Therefore we only need to consider valid linear inequalities of the form $\Psi(s) \geq 1$ where ψ is sublinear. In particular the set $B_\psi = \{x \in W \mid \psi(x - f) \leq 1\}$ contains f in its interior.

Let $K = \{r \in W \mid \psi(r) \leq 1\}$, and let $\hat{K}$ be defined as in (2.3).

Claim 4. *The inequality $\sum_{r \in W} \rho_K(r) s_r \geq 1$ is valid for R_f and $\psi(r) \geq \rho_K(r)$ for all $r \in W$.*

Note that $B_\psi = f + K$. Thus, by Remark 38, $\sum_{r \in W} \rho_K(r) s_r \geq 1$ is valid for R_f. Since ψ is sublinear, it follows from Theorem 15 that $\rho_K(r) \leq \psi(r)$ for every $r \in W$. $\diamond$

By Claim 4, since ρ_K is sublinear, we may assume that $\psi = \rho_K$.

Claim 5. *There exists a valid linear inequality $\Psi'(s) \geq 1$ for R_f dominating $\Psi(s) \geq 1$ such that ψ' is sublinear, $B_{\psi'}$ is a polyhedron, and $\mathrm{rec}(B_{\psi'} \cap V) = \mathrm{lin}(B_{\psi'} \cap V)$.*

Since B_ψ is a lattice-free convex set, it is contained in some maximal lattice-free convex set S by Corollary 33. The set S satisfies one of the statements (i)-(iii) of Theorem 29. By Claim 1, $\mathbf{int}(S) \cap V \neq \emptyset$, hence case (iii) does not apply. Case (ii) does not apply because $\dim(S) = \dim(B_\psi) = \dim(W)$. Therefore case (i) applies. Thus S is a polyhedron and $S \cap V$ is a maximal lattice-free convex set in V. In particular, by Theorem 30, $\mathrm{rec}(S \cap V) = \mathrm{lin}(S \cap V)$. Since S is a polyhedron containing f in its interior, there exists $A \in \mathbb{R}^{t \times q}$ and $b \in \mathbb{R}^t$ such that $b_i > 0$, $i = 1, \ldots, t$, and $S = \{x \in f + W \mid A(x - f) \leq b\}$. Without loss of generality, we may assume that $\sup_{x \in B_\psi} a_i(x - f) = 1$ where a_i denotes the ith row of A, $i = 1, \ldots, t$. By our assumption, $\sup_{r \in K} a_i r = 1$. Therefore $a_i \in K^*$, since $a_i r \leq 1$ for all $r \in K$. Furthermore $a_i \in \mathbf{cl}(\hat{K})$, since $\sup_{r \in K} a_i r = 1$.

Let $\bar{S} = \{x \in f + W \mid A(x - f) \leq e\}$, where e denotes the vector of all ones. Then $B_\psi \subseteq \bar{S} \subseteq S$. Let $Q = \{r \in W \mid Ar \leq e\}$. By Remark 37, $\rho_Q(r) = \max_{i=1,\ldots,t} a_i r$ for all $r \in W$. Since $\bar{S} \subseteq S$, $\bar{S}$ is lattice-free, by Remark 38 the inequality $\sum_{r \in W} \rho_Q(r) s_r \geq 1$ is valid for R_f. Furthermore, since $\{a_1, \ldots, a_t\} \subset \mathbf{cl}(\hat{K})$, by Claim 4 we have

$$\psi(r) = \sup_{y \in \hat{K}} yr \geq \max_{i=1,\ldots,t} a_i r = \rho_Q(r)$$

for all $r \in W$. Let $\psi' = \rho_Q$. Note that $B_{\psi'} = \bar{S}$. So, $\mathrm{rec}(B_{\psi'}) = \mathrm{rec}(\bar{S}) = \{r \in W \mid Ar \leq 0\} = \mathrm{rec}(S)$. Since $\mathrm{rec}(S \cap V) = \mathrm{lin}(S \cap V)$, then $\mathrm{rec}(B_{\psi'} \cap V) = \mathrm{lin}(B_{\psi'} \cap V)$. $\diamond$

By Claim 5, we may assume that $B_\psi = \{x \in f + W \mid A(x - f) \leq e\}$, where $A \in \mathbb{R}^{t \times q}$ and e is the vector of all ones, and that $\mathrm{rec}(B_\psi \cap V) = \mathrm{lin}(B_\psi \cap V)$. Let $a_1, \ldots, a_t$ denote the rows of A. By Claim 4 and Remark 37,

$$\psi(r) = \max_{i=1,\ldots,t} a_i r, \quad \text{for all } r \in W. \tag{3.7}$$

Let G be a matrix such that $W = \{r \in \mathbb{R}^q \mid Gr = 0\}$.

Claim 6. *There exists $\lambda \in \mathbb{R}^\ell$ such that $\psi(r) + \lambda^T Cr \geq 0$ for all $r \in W$.*

Given $\lambda \in \mathbb{R}^\ell$, then by (3.7) $\psi(r) + \lambda^T Cr \geq 0$ for every $r \in W$ if and only if $\min_{r \in W}(\max_{i=1,\ldots,t} a_i r + \lambda^T Cr) = 0$. The latter holds if and only if

$$0 = \min\{z + \lambda^T Cr \mid ez - Ar \geq 0,\ Gr = 0\}.$$

By LP duality, this holds if and only if the following system is feasible

$$\begin{aligned} ey &= 1 \\ A^T y + C^T \lambda - G^T \mu &= 0 \\ y &\geq 0. \end{aligned}$$

Clearly the latter is equivalent to

$$\begin{aligned} A^T y + C^T \lambda - G^T \mu &= 0 \\ y \geq 0,\ y \neq 0. \end{aligned} \tag{3.8}$$

Note that $\mathrm{rec}(B_\psi \cap V) = \{r \in \mathbb{R}^q \mid Ar \leq 0,\ Cr = 0, Gr = 0\}$ and $\mathrm{lin}(B_\psi \cap V) = \{r \in \mathbb{R}^q \mid Ar = 0,\ Cr = 0, Gr = 0\}$. Since $\mathrm{rec}(B_\psi \cap V) = \mathrm{lin}(B_\psi \cap V)$, the system

$$\begin{aligned} Ar &\leq 0 \\ Cr &= 0 \\ Gr &= 0 \\ e^T Ar &= -1 \end{aligned}$$

is infeasible. By Farkas' Lemma (Lemma 20), this is the case if and only if there exists $\gamma \geq 0$, λ, $\tilde{\mu}$, and τ such that

$$A^T \gamma + C^T \lambda + G^T \tilde{\mu} + A^T e\tau = 0, \quad \tau > 0.$$

If we let $y = \gamma + e\tau$ and $\mu = -\tilde{\mu}$, then (y, λ, μ) satisfies (3.8). By the previous argument, λ satisfies the statement of the claim. $\diamond$

Let λ as in Claim 6, and let ψ' be the function defined by $\psi'(r) = \psi(r) + \lambda^T Cr$ for all $r \in W$. So $\psi'(r) \geq 0$ for every $r \in W$. Let $\alpha' = 1 + \lambda^T(d - Cf)$. Then the inequality $\Psi'(s) \geq \alpha'$ is valid for R_f and it is equivalent to $\Psi(s) \geq \alpha$. If $\alpha' \leq 0$, then $\Psi'(s) \geq \alpha'$ is trivial. Thus $\alpha' > 0$. Let $\rho = 1/\alpha'$ and let $\psi'' = \rho\psi'$. Then $\Psi''(s) \geq 1$ is equivalent to $\Psi(s) \geq 1$. By Lemma 35(i), ψ'' is sublinear.

Let B be a maximal lattice-free convex set of $f + W$ containing $B_{\psi''}$. Such a set B exists by Corollary 33.

Claim 7. *$\psi''(r) \geq \psi_B(r)$ for all $r \in W$.*

Let $r \in \mathrm{rec}(B_{\psi''})$. Since ψ'' is nonnegative, $\psi''(r) = 0$. Since $\mathrm{rec}(B_{\psi''}) \subseteq \mathrm{rec}(B)$, $\psi_B(r) \leq 0 = \psi''(r)$. Let $r \notin \mathrm{rec}(B_{\psi''})$. Then $f + \tau r \in \mathbf{bd}(B_{\psi''})$ for some $\tau > 0$, hence $\psi''(\tau r) = 1$ and, by positive homogeneity, $\psi''(r) = \tau^{-1}$. Because $B_{\psi''} \subset B$,

$f + \tau r \in B$. Since $B = \{x \in f + W \mid \psi_B(x - f) \leq 1\}$, it follows that $\psi_B(\tau r) \leq 1$, implying $\psi_B(r) \leq \tau^{-1} = \psi''(r)$. $\diamond$

Claim 7 shows that $\Psi''(s) \geq 1$ is dominated by $\Psi_B(s) \geq 1$, which is minimal by Lemma 39. By Lemma 35(ii), $\Psi(s) \geq 1$ is dominated by a minimal valid linear inequality which is equivalent to $\Psi_B(s) \geq 1$. $\square$

Example. We illustrate the end of the proof in an example. Suppose $W = \{x \in \mathbb{R}^3 \mid x_2 + \sqrt{2}x_3 = 0\}$, and let $f = (\frac{1}{2}, 0, 0)$. Note that $f + W = W$. All integral points in W are of the form $(k, 0, 0)$, $k \in \mathbb{Z}$, hence $V = \{x \in W \mid x_2 = 0\}$. Thus $\mathcal{V} = \{s \in \mathcal{W} \mid \sum_{r \in W} r_2 s_r = 0\}$.

Consider the function $\psi : W \to \mathbb{R}$ defined by $\psi(r) = \max\{-4r_1 - 4r_2, 4r_1 - 4r_2\}$. The set $B_\psi = \{x \in W \mid -4(x_1 - \frac{1}{2}) - 4x_2 \leq 1,\ 4(x_1 - \frac{1}{2}) - 4x_2 \leq 1\}$ does not contain any integer point, hence $\Psi(s) \geq 1$ is valid for R_f. Note that B_ψ is not maximal (see Figure 3.2).

Given $\lambda = 4$, let $\psi'(r) = \psi(r) + \lambda r_2$ for all $r \in W$. Note that $\psi'(r) = \max\{-4r_1, 4r_1\} \geq 0$ for all $r \in W$. The set $B_{\psi'} = \{x \in W \mid -4(x_1 - \frac{1}{2}) \leq 1,\ 4(x_1 - \frac{1}{2}) \leq 1\}$ is contained in the maximal lattice-free convex set $B = \{x \in W \mid -2(x_1 - \frac{1}{2}) \leq 1,\ 2(x_1 - \frac{1}{2}) \leq 1\}$, hence ψ' is pointwise larger than the function ψ_B defined by $\psi_B(r) = \max\{-2r_1, 2r_1\}$ and $\Psi_B(s) \geq 1$ is valid for R_f.

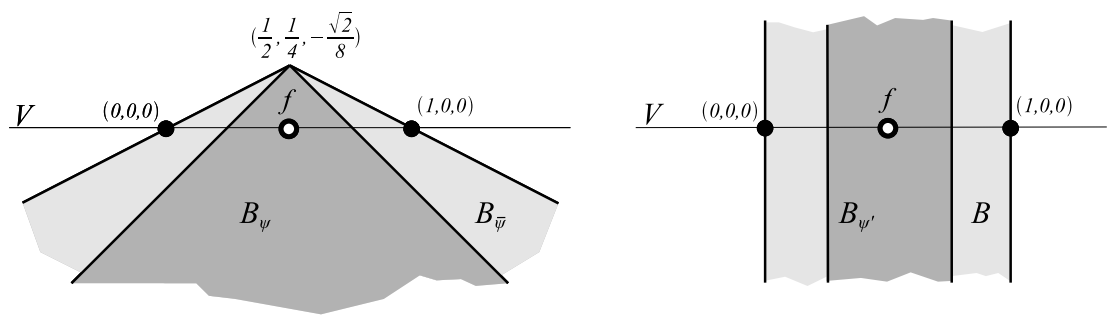


Figure 3.2: Lattice-free sets in the 2-dimensional space W.

By construction, the function $\bar{\psi}$ defined by $\psi_B(r) - \lambda r_2$ for all $r \in W$ is pointwise smaller than ψ and $\bar{\Psi}(s) \geq 1$ is valid for R_f. Moreover, $B_{\bar{\psi}} = \{x \in W \mid -2(x_1 - \frac{1}{2}) - 4x_1 \leq 1,\ 2(x_1 - \frac{1}{2}) - 4x_1 \leq 1\}$ is a maximal lattice-free convex set containing B_ψ. Note that the recession cones of B_ψ and $B_{\bar{\psi}}$ are full dimensional, hence ψ and $\bar{\psi}$ take negative values on elements of the recession cone. For example $\psi(0, -1, \frac{1}{\sqrt{2}}) =$

$\bar{\psi}(0,-1,\frac{1}{\sqrt{2}}) = -4$. The recession cones of $B_{\psi'}$ and B coincide and are not full dimensional, thus $\psi'(0,-1,\frac{1}{\sqrt{2}}) = \psi_B(0,-1,\frac{1}{\sqrt{2}}) = 0$, since the vector $(0,-1,\frac{1}{\sqrt{2}})$ is in the recession cone of B.

3.1.3 Intersection of all Minimal Inequalities

In this section we prove Theorem 27. First we need the following.

Lemma 40. *Let $\psi : W \to \mathbb{R}$ be a continuous function that is positively homogeneous. Then the linear functional $\Psi : \mathcal{W} \to \mathbb{R}$, defined by $\Psi(s) = \sum_{r\in W} \psi(r)s_r$, is continuous with respect to $(\mathcal{W}, \|\cdot\|_H)$.*

Proof: Define $\gamma = \sup\{|\psi(r)| : r \in W, \|r\| = 1\}$. Since the set $\{r \in R_f(W) : \|r\| = 1\}$ is compact and ψ is continuous, γ is well defined (that is, it is finite). Given $s, s' \in \mathcal{W}$, we will show $|\Psi(s') - \Psi(s)| \leq \gamma\|s' - s\|_H$, which implies that Ψ is continuous. Indeed

$$\begin{aligned}
|\Psi(s') - \Psi(s)| &= |\sum_{r\in W} \psi(r)(s'_r - s_r)| \\
&\leq \sum_{r\in W} |\psi(r)|\,|s'_r - s_r| \\
&= \sum_{r\in W\,:\,\|r\|=1} \sum_{\alpha>0} |\psi(\alpha r)|\,|s'_{\alpha r} - s_{\alpha r}| \\
&= \sum_{r\in W\,:\,\|r\|=1} |\psi(r)| \sum_{\alpha>0} \alpha|s'_{\alpha r} - s_{\alpha r}| \quad \text{(by positive homogeneity of } \psi) \\
&\leq \gamma \sum_{r\in W\,:\,\|r\|=1} \sum_{\alpha>0} \alpha|s'_{\alpha r} - s_{\alpha r}| \\
&= \gamma \sum_{r\in W\setminus\{0\}} \|r\||s'_r - s_r| \quad \leq \gamma\|s' - s\|_H
\end{aligned}$$

□

Proof of Theorem 27. "$\subseteq$" By Lemma 40, Ψ_B is continuous in $(\mathcal{W}, \|\cdot\|_H)$ for every $B \in \mathcal{B}_W$, therefore $\{s \in \mathcal{W} : \Psi_B(s) \geq 1\}$ is a closed half-space of $(\mathcal{W}, \|\cdot\|_H)$. It is immediate to show that also $\{s \in \mathcal{W} : s_r \geq 0, r \in W\}$ is a closed set in $(\mathcal{W}, \|\cdot\|_H)$.

Since $\mathcal{V} = \{s \in \mathcal{W} \mid \sum_{r \in W}(Cr)s_r = d - Cf\}$, and since for each row c_i of C the function $r \mapsto c_i r$ is positive homogeneous, then by Lemma 40 $\mathcal{V}$ is also closed. Thus

$$\{s \in \mathcal{V} \,:\, \Psi_B(s) \geq 1, B \in \mathcal{B}_W;\ s_r \geq 0,\, r \in W\}$$

is an intersection of closed sets, and is therefore a closed set of $(\mathcal{W}, \|\cdot\|_H)$. Thus, since it contains $\mathrm{conv}(R_f(W))$, it also contains $\overline{\mathrm{conv}}(R_f(W))$.

"$\supseteq$" We only need to show that, for every $\bar{s} \in \mathcal{V}$ such that $\bar{s} \notin \overline{\mathrm{conv}}(R_f(W))$ and $\bar{s}_r \geq 0$ for every $r \in W$, there exists $B \in \mathcal{B}_W$ such that $\sum_{r \in W} \psi_B(r)\bar{s}_r < 1$.

By Theorem 12, there exists a linear functional $\Psi \,:\, \mathcal{W} \to \mathbb{R}$ such that $\Psi(\bar{s}) < \alpha$ and $\Psi(s) \geq \alpha$ for every $s \in \overline{\mathrm{conv}}(R_f)$. By the first part of Theorem 26, we may assume that $\Psi(s) \geq \alpha$ is a nontrivial minimal valid linear inequality. By the second part of Theorem 26, this inequality is equivalent to an inequality of the form $\sum_{r \in W} \psi_B(r)s_r \geq 1$ for some maximal lattice-free convex set B of W with f in its interior. □

3.2 Constrained Basic Variables and Maximal S-free Convex Sets

In this section, we study a generalization of (2.13). Recall that (2.13) was obtained from the corner polyhedron by dropping the integrality constraints on the non-basic variables. The corner polyhedron, in turn, relaxed the non-negativity constraints on the basic variables. In this section, we will reinstate the non-negativity constraints on the integer basic variables. In fact, we will impose the constraints $x \in S$, where S is the set of integral points in some given rational polyhedron such that $\dim(S) = n$. A polyhedron $Ax \leq b$ is *rational* if the matrix A and vector b have rational entries. Formally, we will study the following generalization of (2.13) introduced first by Johnson [49].

$$\begin{aligned} x &= f + \sum_{r \in \mathbb{R}^n} r s_r \\ x &\in S \\ & \quad s_r \geq 0, \quad r \in \mathbb{R}^n \\ & \quad s \text{ has finite support} \end{aligned} \tag{3.9}$$

where $f \in \mathrm{conv}(S) \setminus \mathbb{Z}^n$.

Let $R_{f,S}$ be the set of all infinite dimensional vectors $s = (s_r)_{r \in \mathbb{R}^n}$ such that (3.9) is satisfied. Although in this model we have constraints on the basic variables, it still assumes that the non-basic variables are continuous. Therefore, it is not a generalization of the corner polyhedron. However, it does generalize (2.13) and we will be able to obtain theorems showing the correspondence between valid inequalities for (3.9) and maximal convex sets of a more general kind (in relation to lattice-free convex sets), called the S-free convex sets.

We can define the notion of valid inequalities or functions as described in Section 3.1 for our new model in an analogous manner. The motivation to study this infinite model and valid inequalities for it is the same as outlined in Section 3.1 : we simply need to restrict the valid function to the given set of finite rays to obtain valid inequalities for the finite problem. In this Section, we will be only concentrating on valid inequalities which cut off the current infeasible solution $x = f$, $s = 0$. Therefore, unlike in Section 3.1, we only need to consider 1 as the right hand side in our valid inequalities. In other words, for the purposes of this section, a function $\psi : \mathbb{R}^n \to \mathbb{R}$ is *valid* (with respect to f and S) if the linear inequality

$$\sum_{r \in \mathbb{R}^n} \psi(r) s_r \geq 1 \tag{3.10}$$

is satisfied by every $s \in R_{f,S}$.

Dey And Wolsey [38] introduce the notion of *S-free convex set* as a convex set without points of S in its interior, and show the connection between valid functions and S-free convex sets with f in their interior.

Using the same ideas as in the previous sections, we can define the notion of a *minimal* valid function and show that every valid function is dominated by a minimal valid function. One can also show that minimal valid functions are sublinear. This fact was also shown by Dey and Wolsey [38] and furthermore they show that B_ψ is an S-free convex set with f in its interior. B_ψ is defined analogously to the definition in Section 3.1.2.

$$B_\psi = \{x \in \mathbb{R}^n \mid \psi(x - f) \leq 1\} \tag{3.11}$$

In this section, we prove that if ψ is a minimal valid function, then B_ψ is a *maximal* S-free convex set. For this purpose, we first show in Section 3.2.1 that maximal S-free convex sets are polyhedra. Therefore a maximal S-free convex set

$B \subseteq \mathbb{R}^n$ containing f in its interior can be uniquely written in the form $B = \{x \in \mathbb{R}^n : \ a_i(x - f) \leq 1, \ i = 1, \ldots, k\}$. Let $\psi_B : \mathbb{R}^n \to \mathbb{R}$ be the function defined by

$$\psi_B(r) = \max_{i=1,\ldots,k} a_i r, \quad \forall r \in \mathbb{R}^n. \tag{3.12}$$

The following remark is easy to verify.

Remark 41. *ψ_B is sublinear and $B = \{x \in \mathbb{R}^n \mid \psi_B(x - f) \leq 1\}$.*

In Section 3.2.2 we will prove that every minimal valid function is of the form ψ_B for some maximal S-free convex set B containing f in its interior. Conversely, if B is a maximal S-free convex set containing f in its interior, then ψ_B is a minimal valid function.

3.2.1 Maximal S-free Convex Sets

Let $S \subseteq \mathbb{Z}^n$ be the set of integral points in some rational polyhedron of $\mathbb{R}^n$. We say that $B \subset \mathbb{R}^n$ is an *S-free convex set* if B is convex and does not contain any point of S in its interior. We say that B is a *maximal S-free convex set* if it is an S-free convex set and it is not properly contained in any S-free convex set. It follows from Zorn's lemma that every S-free convex set is contained in a maximal S-free convex set.

When $S = \mathbb{Z}^n$, an S-free convex set is a lattice-free convex set and Theorem 25 characterizes their structure. The next theorem characterizes maximal S-free convex sets.

Theorem 42. *Let S be the set of integral points in some rational polyhedron of $\mathbb{R}^n$ such that $\dim(S) = n$. A set $B \subset \mathbb{R}^n$ is a maximal S-free convex set if and only if one of the following holds:*

(i) *B is a polyhedron such that $B \cap \operatorname{conv}(S)$ has nonempty interior, B does not contain any point of S in its interior and there is a point of S in the relative interior of each of its facets.*

(ii) *B is a half-space of $\mathbb{R}^n$ such that $B \cap \operatorname{conv}(S)$ has empty interior and the boundary of B is a supporting hyperplane of $\operatorname{conv}(S)$.*

(iii) *B is a hyperplane of $\mathbb{R}^n$ such that $\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$ is not rational.*

Furthermore, if (i) holds, the recession cone of $B \cap \mathrm{conv}(S)$ is rational and it is contained in the lineality space of B.

We illustrate case (i) of the theorem in the plane in Figure 3.3. The question of the polyhedrality of maximal S-free convex sets was raised by Dey and Wolsey [38]. They proved that this is the case for a maximal S-free convex set B, under the assumptions that $B \cap \mathrm{conv}(S)$ has nonempty interior and that the recession cone of $B \cap \mathrm{conv}(S)$ is finitely generated and rational. Theorem 42 settles the question in general.

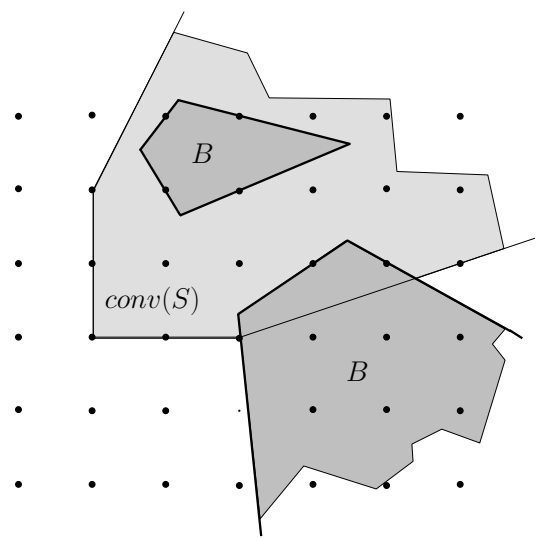


Figure 3.3: Two examples of S-free sets in the plane (case (i) of Theorem 42). The light gray region indicates $\mathrm{conv}(S)$ and the dark grey regions illustrate the S-free sets. A jagged line indicates that the region extends to infinity.

To prove Theorem 42 we will need the following lemma.

Lemma 43. *Let B be an S-free convex set such that $B \cap \mathrm{conv}(S)$ has nonempty interior. For every $r \in \mathrm{rec}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$, $B + \langle r \rangle$ is S-free.*

Proof. Let $C = \mathrm{rec}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$ and $r \in C \setminus \{0\}$. Suppose by contradiction that there exists $y \in S \cap \mathbf{int}(B + \langle r \rangle)$. We show that $y \in \mathbf{int}(B) + \langle r \rangle$. If not, $(y + \langle r \rangle) \cap \mathbf{int}(B) = \emptyset$, which implies that there is a hyperplane H separating the line $y + \langle r \rangle$ and $B + \langle r \rangle$, a contradiction. Thus there exists $\bar{\lambda}$ such that $\bar{y} = y + \bar{\lambda} r \in \mathbf{int}(B)$, i.e. there exists $\varepsilon > 0$ such that B contains the open ball $B_\varepsilon(\bar{y})$ of radius ε centered

at $\bar{y}$. Since $r \in C \subseteq \mathrm{rec}(B)$, it follows that $B_\varepsilon(\bar{y}) + \{\lambda r \,|\, \lambda \geq 0\} \subset B$. Since $y \in \mathbb{Z}^n$, by Lemma 7 there exists $z \in \mathbb{Z}^n$ at distance less than ε from the half line $\{y + \lambda r \,|\, \lambda \geq \bar{\lambda}\}$. Thus $z \in B_\varepsilon(\bar{y}) + \{\lambda r \,|\, \lambda \geq 0\}$, hence $z \in \mathbf{int}(B)$. Note that the half-line $\{y + \lambda r \,|\, \lambda \geq \bar{\lambda}\}$ is in $\mathrm{conv}(S)$, since $y \in S$ and $r \in \mathrm{rec}(\mathrm{conv}(S))$. Since $\mathrm{conv}(S)$ is a rational polyhedron, for $\varepsilon > 0$ sufficiently small every integral point at distance at most ε from $\mathrm{conv}(S)$ is in $\mathrm{conv}(S)$. Therefore $z \in S$, a contradiction. $\square$

Proof of Theorem 42. The proof of the "if" part is similar to the proof for the lattice-free case (see Proof of Theorem 25). We show the "only if" part. Let B be a maximal S-free convex set. If $\dim(B) < n$, then B is contained in some affine hyperplane K. Since K has empty interior, K is S-free, thus $B = K$ by maximality of B. Next we show that $\mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$ is not rational. Suppose not. Then the linear subspace $L = \langle \mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S)) \rangle$ is rational. Therefore the projection Λ of $\mathbb{Z}^n$ onto $L^\perp$ is a lattice of $L^\perp$ (see, for example, Barvinok [17] p 284 problem 3). The projection S' of S onto $L^\perp$ is a subset of Λ. Let B' be the projection of B onto $L^\perp$. Then $B' \cap \mathrm{conv}(S')$ is the projection of $B \cap \mathrm{conv}(S)$ onto $L^\perp$. Since B is a hyperplane, $\mathrm{lin}(B) = \mathrm{rec}(B)$. This implies that $B' \cap \mathrm{conv}(S')$ is bounded : otherwise there is an unbounded direction $d \in L^\perp$ in $\mathrm{rec}(B') \cap \mathrm{rec}(\mathrm{conv}(S'))$ and so $d + l \in \mathrm{rec}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$ for some $l \in L$. Since $\mathrm{rec}(B) \cap \mathrm{rec}(\mathrm{conv}(S)) = \mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$, this would imply that $d \in L$ which is a contradiction. Fix $\delta > 0$. Since Λ is a lattice and $S' \subseteq \Lambda$, there is a finite number of points at distance less than δ from the bounded set $B' \cap \mathrm{conv}(S')$ in $L^\perp$. It follows that there exists $\varepsilon > 0$ such that every point of S' has distance at least ε from $B' \cap \mathrm{conv}(S')$. Let $B'' = \{v + w \,|\, v \in B,\, w \in L^\perp, \|w\| \leq \varepsilon\}$. The set B'' is S-free by the choice of ε, but B'' strictly contains B, contradicting the maximality of B. Therefore (iii) holds when $\dim(B) < n$. Hence we may assume $\dim(B) = n$. If $B \cap \mathrm{conv}(S)$ has empty interior, then there exists a hyperplane separating B and $\mathrm{conv}(S)$ which is supporting for $\mathrm{conv}(S)$. By maximality of B case (ii) follows.

Therefore we may assume that $B \cap \mathrm{conv}(S)$ has nonempty interior. We show that B satisfies (i).

Claim 8. *There exists a rational polyhedron P such that:*
$i)$ $\mathrm{conv}(S) \subset \mathbf{int}(P)$,
$ii)$ *The set $K = B \cap P$ is lattice-free,*
$iii)$ *For every facet F of P, $F \cap K$ is a facet of K,*
$iv)$ *For every facet F of P, $F \cap K$ contains an integral point in its relative interior.*

Since $\mathrm{conv}(S)$ is a rational polyhedron, there exist integral A and b such that $\mathrm{conv}(S) = \{x \in \mathbb{R}^n \,|\, Ax \leq b\}$. The set $P' = \{x \in \mathbb{R}^n \,|\, Ax \leq b + \frac{1}{2}\mathbf{1}\}$ satisfies $i)$.

The set $B \cap P'$ is lattice-free since B is S-free and P' does not contain any point in $\mathbb{Z}^n \setminus S$, thus P' also satisfies $ii)$. Let $\bar{A}x \leq \bar{b}$ be the system containing all inequalities of $Ax \leq b + \frac{1}{2}\mathbf{1}$ that define facets of $B \cap P'$. Let $P_0 = \{x \in \mathbb{R}^n \mid \bar{A}x \leq \bar{b}\}$. Then P_0 satisfies $i), ii), iii)$. See Figure 3.4 for an illustration.

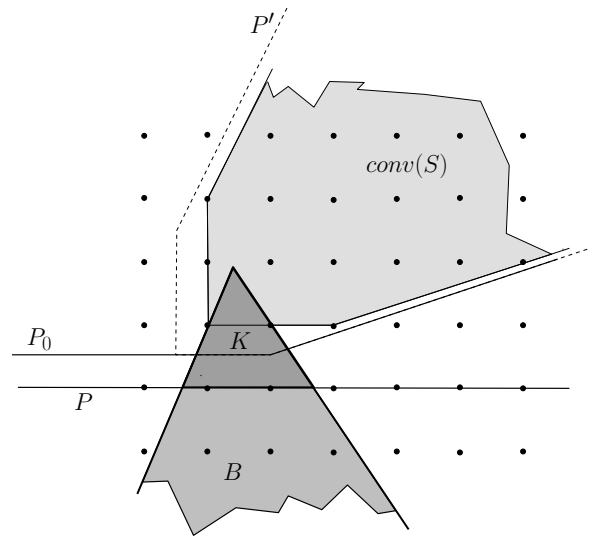


Figure 3.4: Illustration for Claim 8.

It will be more convenient to write P_0 as intersection of the half-spaces defining the facets of P_0, $P_0 = \cap_{H \in \mathcal{F}_0} H$. We construct a sequence of rational polyhedra $P_0 \subset P_1 \subset \ldots \subset P_t$ such that P_i satisfies $i), ii), iii)$, $i = 1, \ldots, t$, and such that P_t satisfies $iv)$. Given P_i, we construct P_{i+1} as follows. Let $P_i = \cap_{H \in \mathcal{F}_i} H$, where $\mathcal{F}_i$ is the set of half spaces defining facets of P_i. Let $\bar{H}$ be a half-space in $\mathcal{F}_i$ defining a facet of $B \cap P_i$ that does not contain an integral point in its relative interior; if no such $\bar{H}$ exists, then P_i satisfies $iv)$ and we are done. If $B \cap \bigcap_{H \in \mathcal{F}_i \setminus \{\bar{H}\}} H$ does not contain any integral point in its interior, let $\mathcal{F}_{i+1} = \mathcal{F}_i \setminus \{\bar{H}\}$. Otherwise, since P_i is rational, among all integral points in the interior of $B \cap \bigcap_{H \in \mathcal{F}_i \setminus \{\bar{H}\}} H$ there exists one, say $\bar{x}$, at minimum distance from $\bar{H}$. Let H' be the half-space containing $\bar{H}$ with $\bar{x}$ on its boundary. Let $\mathcal{F}_{i+1} = \mathcal{F}_i \setminus \{\bar{H}\} \cap \{H'\}$. Observe that H' defines a facet of P_{i+1} since $\bar{x}$ is in the interior of $B \cap \bigcap_{H \in \mathcal{F}_{i+1} \setminus \{H'\}} H$ and it is on the boundary of H'. So $i), ii), iii)$ are satisfied and P_{i+1} has fewer facets that violate $iv)$ than P_i. $\diamond$

Let T be a maximal lattice-free convex set containing the set K defined in Claim 8. As remarked earlier, such a set T exists. By Theorem 30, T is a polyhedron with an integral point in the relative interior of each of its facets. Let H be

a hyperplane that defines a facet of P. Since $K \cap H$ is a facet of K with an integral point in its relative interior, it follows that H defines a facet of T. This implies that $T \subset P$. Therefore we can write T as

$$T = P \cap \bigcap_{i=1}^{k} H_i, \tag{3.13}$$

where H_i are halfspaces. Let $\bar{H}_i = \mathbb{R}^n \setminus \mathbf{int}(H_i)$, $i = 1, \ldots, k$.

Claim 9. *B is a polyhedron.*

We first show that, for $i = 1, \ldots, k$, $\mathbf{int}(B) \cap (\bar{H}_i \cap \mathrm{conv}(S)) = \emptyset$. Consider $y \in \mathbf{int}(B) \cap \bar{H}_i$. Since $y \in \bar{H}_i$ and K is contained in T, $y \notin \mathbf{int}(K)$. Since $K = B \cap P$ and $y \in \mathbf{int}(B) \setminus \mathbf{int}(K)$, it follows that $y \notin \mathbf{int}(P)$. Hence $y \notin \mathrm{conv}(S)$ because $\mathrm{conv}(S) \subseteq \mathbf{int}(P)$.

Thus, for $i = 1, \ldots, k$, there exists a hyperplane separating B and $\bar{H}_i \cap \mathrm{conv}(S)$. Hence there exists a halfspace K_i such that $B \subset K_i$ and $\bar{H}_i \cap \mathrm{conv}(S)$ is disjoint from the interior of K_i. We claim that the set $B' = \cap_{i=1}^{k} K_i$ is S-free. Indeed, let $y \in S$. Then y is not interior of T. Since $y \in \mathrm{conv}(S)$ and $\mathrm{conv}(S) \subseteq \mathbf{int}(P)$, y is in the interior of P. Hence, by (3.13), there exists $i \in \{1, \ldots, k\}$ such that y is not in the interior of H_i. Thus $y \in \bar{H}_i \cap \mathrm{conv}(S)$. By construction, y is not in the interior of K_i, hence y is not in the interior of B'. Thus B' is an S-free convex set containing B. Since B is maximal, $B' = B$. $\diamond$

Claim 10. $\mathrm{lin}(K) = \mathrm{rec}(K)$.

Let $r \in \mathrm{rec}(K)$. We show $-r \in \mathrm{rec}(K)$. By Lemma 43 applied to $\mathbb{Z}^n$, $K + \langle r \rangle$ is lattice-free. We observe that $B + \langle r \rangle$ is S-free. If not, let $y \in S \cap \mathbf{int}(B + \langle r \rangle)$. Since $S \subseteq \mathbf{int}(P)$, $y \in \mathbf{int}(P + \langle r \rangle)$, hence $y \in \mathbf{int}(K + \langle r \rangle)$, a contradiction. Hence, by maximality of B, $B = B + \langle r \rangle$. Thus $-r \in \mathrm{rec}(B)$. Suppose that $-r \notin \mathrm{rec}(P)$. Then there exists a facet F of P that is not parallel to r. By construction, $F \cap K$ is a facet of K containing an integral point $\bar{x}$ in its relative interior. The point $\bar{x}$ is then in the interior of $K + \langle r \rangle$, a contradiction. $\diamond$

Claim 11. $\mathrm{lin}(K)$ *is rational.*

Consider the maximal lattice-free convex set T containing K considered earlier. By Theorem 30, $\mathrm{lin}(T) = \mathrm{rec}(T)$, and $\mathrm{lin}(T)$ is rational. Clearly $\mathrm{lin}(T) \supseteq \mathrm{lin}(K)$. Hence, if the claim does not hold, there exists a rational vector $r \in \mathrm{lin}(T) \setminus \mathrm{lin}(K)$. By (3.13), $r \in \mathrm{lin}(P)$. Since $K = B \cap P$, $r \notin \mathrm{lin}(B)$. Hence $B \subset B + \langle r \rangle$. We

will show that $B + \langle r \rangle$ is S-free, contradicting the maximality of B. Suppose there exists $y \in S \cap \mathbf{int}(B + \langle r \rangle)$. Since $\mathrm{conv}(S) \subseteq \mathbf{int}(P)$, $y \in \mathbf{int}(P) \subseteq \mathbf{int}(P) + \langle r \rangle$. Therefore $y \in \mathbf{int}(B \cap P) + \langle r \rangle$. Since $B \cap P \subseteq T$, then $y \in \mathbf{int}(T) + \langle r \rangle = \mathbf{int}(T)$ where the last equality follows from $r \in \mathrm{lin}(T)$. This contradicts the fact that T is lattice-free. $\diamond$

By Lemma 43 and by the maximality of B, $\mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S)) = \mathrm{rec}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$.

Claim 12. $\mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$ *is rational.*

Since $\mathrm{lin}(K)$ and $\mathrm{rec}(\mathrm{conv}(S))$ are both rational, we only need to show $\mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S)) = \mathrm{lin}(K) \cap \mathrm{rec}(\mathrm{conv}(S))$. The "$\supseteq$" direction follows from $B \supseteq K$. For the other direction, note that, since $\mathrm{conv}(S) \subseteq P$, we have $\mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S)) \subseteq \mathrm{lin}(B) \cap \mathrm{rec}(P) = \mathrm{lin}(B \cap P) = \mathrm{lin}(K)$, hence $\mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S)) \subseteq \mathrm{lin}(K) \cap \mathrm{rec}(\mathrm{conv}(S))$. $\diamond$

Claim 13. *Every facet of B contains a point of S in its relative interior.*

Let L be the linear space generated by $\mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$. By Claim 12, L is rational. Let B', S', Λ be the projections of B, S, $\mathbb{Z}^n$, respectively, onto $L^\perp$. Since L is rational, Λ is a lattice of $L^\perp$ and $S' = \mathrm{conv}(S') \cap \Lambda$. Also, B' is a maximal S'-free convex set of $L^\perp$, since for any S'-free set D of $L^\perp$, $D + L$ is S-free. Note that $\mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S)) = \mathrm{rec}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$ implies that $B' \cap \mathrm{conv}(S')$ is bounded. Otherwise there is an unbounded direction $d \in L^\perp$ in $\mathrm{rec}(B') \cap \mathrm{rec}(\mathrm{conv}(S'))$ and so $d + l \in \mathrm{rec}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$ for some $l \in L$. Since $\mathrm{rec}(B) \cap \mathrm{rec}(\mathrm{conv}(S)) = \mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$, this would imply that $d \in L$ which is a contradiction. Let $B' = \{x \in L^\perp \mid \alpha_i x \leq \beta_i,\ i = 1, \ldots, t\}$. Given $\varepsilon > 0$, let $\bar{B} = \{x \in L^\perp \mid \alpha_i x \leq \beta_i,\ i = 1, \ldots, t-1,\ \alpha_t x \leq \beta_t + \varepsilon\}$. The polyhedron $\mathrm{conv}(S') \cap \bar{B}$ is a polytope since it has the same recession cone as $\mathrm{conv}(S') \cap B'$. The polytope $\mathrm{conv}(S') \cap \bar{B}$ contains points of S' in its interior by the maximality of B'. Since Λ is a lattice of $L^\perp$, $\mathbf{int}(\mathrm{conv}(S') \cap \bar{B})$ has a finite number of points in S', hence there exists one minimizing $\alpha_t x$, say z. By construction, the polyhedron $B'' = \{x \in L^\perp \mid \alpha_i x \leq \beta_i,\ i = 1, \ldots, t-1,\ \alpha_t x \leq \alpha_t z\}$ does not contain any point of S' in its interior and contains B'. By the maximality of B', $B' = B''$ hence B' contains z in its relative interior, and B contains a point of S in its relative interior. $\square$

3.2.2 Minimal Valid Functions

In this section, we prove the following correspondence between minimal valid inequalities and maximal S-free sets.

Theorem 44. *For every minimal valid function ψ, there exists a maximal S-free convex set B with f in its interior such that ψ_B dominates ψ. Furthermore, if B is a maximal S-free convex set containing f in its interior, then ψ_B is a minimal valid function.*

We will need the following lemma whose proof is the same as the proof of Lemma 34.

Lemma 45. *Every valid function is dominated by a sublinear valid function.*

Given a valid sublinear function ψ, the set $B_\psi = \{x \in \mathbb{R}^n \,|\, \psi(x-f) \leq 1\}$ is closed, convex, and contains f in its interior. Since ψ is a valid function, B_ψ is S-free. Indeed the interior of B_ψ is $\mathbf{int}(B_\psi) = \{x \in \mathbb{R}^n : \psi(x-f) < 1\}$. Its boundary is $\mathbf{bd}(B_\psi) = \{x \in \mathbb{R}^n : \psi(x-f) = 1\}$, and its recession cone is $\mathrm{rec}(B_\psi) = \{x \in \mathbb{R}^n : \psi(x-f) \leq 0\}$.

We will again need the sublinear function ρ_K associated with a closed convex set K with the origin in its interior defined as in (3.6). The following remark is analogous to Remark 38 and the previous proof easily adapts to the S-free case.

Remark 46. *Let B be a closed S-free convex set in $\mathbb{R}^n$ with f in its interior, and let $K = B - f$. Then ρ_K is a valid function.*

Lemma 47. *Let C be a closed S-free convex set containing f in its interior, and let $K = C - f$. There exists a maximal S-free convex set $B = \{x \in \mathbb{R}^n \,|\, a_i(x-f) \leq 1,\ i = 1, \ldots, k\}$ such that $a_i \in \mathbf{cl}(\mathrm{conv}(\hat{K}))$ for $i = 1, \ldots, k$.*

Proof. Since C is an S-free convex set, it is contained in some maximal S-free convex set T. The set T satisfies one of the statements (i)-(iii) of Theorem 42. By assumption, $f \in \mathrm{conv}(S)$ and f is in the interior of C. Since $\dim(S) = n$, $\mathrm{conv}(S)$ is a full dimensional polyhedron, thus $\mathbf{int}(C \cap \mathrm{conv}(S)) \neq \emptyset$. This implies that $\mathbf{int}(T \cap \mathrm{conv}(S)) \neq \emptyset$, hence case (i) applies.

Thus T is a polyhedron and $\mathrm{rec}(T \cap \mathrm{conv}(S)) = \mathrm{lin}(T) \cap \mathrm{rec}(\mathrm{conv}(S))$ is rational. Let us choose T such that the dimension of $\mathrm{lin}(T)$ is largest possible.

Since T is a polyhedron containing f in its interior, there exists $D \in \mathbb{R}^{t\times q}$ and $b \in \mathbb{R}^t$ such that $b_i > 0$, $i = 1, \ldots, t$, and $T = \{x \in \mathbb{R}^n \mid D(x - f) \leq b\}$. Without loss of generality, we may assume that $\sup_{x\in C} d_i(x - f) = 1$ where d_i denotes the ith row of D, $i = 1, \ldots, t$. By our assumption, $\sup_{r\in K} d_i r = 1$. Therefore $d_i \in K^*$, since $d_i r \leq 1$ for all $r \in K$. Furthermore $d_i \in \mathbf{cl}(\hat{K})$, since $\sup_{r\in K} d_i r = 1$.

Let $P = \{x \in \mathbb{R}^n \mid D(x - f) \leq e\}$. Note that $\mathrm{lin}(P) = \mathrm{lin}(T)$. By our choice of T, $P + \langle r\rangle$ is not S-free for any $r \in \mathrm{rec}(\mathrm{conv}(S)) \setminus \mathrm{lin}(P)$, otherwise P would be contained in a maximal S-free convex set whose lineality space contains $\mathrm{lin}(T)+\langle r\rangle$, a contradiction.

Let $L = \langle \mathrm{rec}(P \cap \mathrm{conv}(S))\rangle$. Since $\mathrm{lin}(P) = \mathrm{lin}(T)$, L is a rational space. Note that $L \subseteq \mathrm{lin}(P)$, implying that $d_i \in L^\perp$ for $i = 1, \ldots, t$.

We observe next that we may assume that $P\cap\mathrm{conv}(S)$ is bounded. Indeed, let $\bar{P}$, $\bar{S}$, Λ be the projections onto $L^\perp$ of P, S, and $\mathbb{Z}^n$, respectively. Since L is a rational space, Λ is a lattice of $L^\perp$ and $\bar{S} = \mathrm{conv}(\bar{S}) \cap \Lambda$. Note that $\bar{P} \cap \mathrm{conv}(\bar{S})$ is bounded, since $L \supseteq \mathrm{rec}(P \cap \mathrm{conv}(S))$. If we are given a maximal $\bar{S}$-free convex set $\bar{B}$ in $L^\perp$ such that $\bar{B} = \{x \in L^\perp \mid a_i(x - f) \leq 1,\ i = 1, \ldots, h\}$ and $a_i \in \mathrm{conv}\{d_1, \ldots, d_t\}$ for $i = 1, \ldots, h$, then $B = \bar{B} + L$ is the set $B = \{x \in \mathbb{R}^n \mid a_i(x - f) \leq 1,\ i = 1, \ldots, h\}$. Since $\bar{B}$ contains a point of $\bar{S}$ in the relative interior of each of its facets, B contains a point of S in the relative interior of each of its facets, thus B is a maximal S-free convex set.

Thus we assume that $P \cap \mathrm{conv}(S)$ is bounded, so $\dim(L) = 0$. If all facets of P contain a point of S in their relative interior, then P is a maximal S-free convex set, thus the statement of the lemma holds. Otherwise we describe a procedure that replaces one of the inequalities defining a facet of P without any point of S in its relative interior with an inequality which is a convex combination of the inequalities of $D(x - f) \leq e$, such that the new polyhedron thus obtained is S-free and has one fewer facet without points of S in its relative interior. More formally, suppose the facet of P defined by $d_1(x - f) \leq 1$ does not contain any point of S in its relative interior. Given $\lambda \in [0, 1]$, let

$$P(\lambda) = \{x \in \mathbb{R}^n \mid [\lambda d_1 + (1 - \lambda)d_2](x - f) \leq 1, \quad d_i(x - f) \leq 1 \ \ i = 2, \ldots, t\}.$$

Note that $P(1) = P$ and $P(0)$ is obtained from P by removing the inequality $d_1(x - f) \leq 1$. Furthermore, given $0 \leq \lambda' \leq \lambda'' \leq 1$, we have $P(\lambda') \supseteq P(\lambda'')$.

Let $r_1, \ldots, r_m$ be generators of $\mathrm{rec}(\mathrm{conv}(S))$. Note that, since $P \cap \mathrm{conv}(S)$ is bounded, for every $j = 1, \ldots, m$ there exists $i \in \{1, \ldots, t\}$ such that $d_i r_j > 0$. Let

$r_1, \ldots, r_h$ be the generators of $\mathrm{rec}(\mathrm{conv}(S))$ satisfying

$$\begin{aligned} d_1 r_j &> 0 \\ d_i r_j &\leq 0 \qquad i = 2, \ldots, t. \end{aligned}$$

Note that, if no such generators exist, then $P(0) \cap \mathrm{conv}(S)$ is bounded. Otherwise $P(\lambda) \cap \mathrm{conv}(S)$ is bounded if and only if, for $j = 1, \ldots, h$

$$[\lambda d_1 + (1-\lambda) d_2] r_j > 0.$$

This is the case if and only if $\lambda > \lambda^*$, where

$$\lambda^* = \max_{j=1,\ldots,h} \frac{-d_2 r_j}{(d_1 - d_2) r_j}.$$

Let r^* be one of the vectors $r_1, \ldots, r_h$ attaining the maximum in the previous equation. Then $r^* \in \mathrm{rec}(P(\lambda^* \cap \mathrm{conv}(S)$.

Note that $P(\lambda^*)$ is not S-free otherwise $P(\lambda^*) + \langle r^* \rangle$ is S-free by Lemma 43, and so is $P + \langle r^* \rangle$, a contradiction.

Thus $P(\lambda^*)$ contains a point of S in its interior. That is, there exists a point $\bar{x} \in S$ such that $[\lambda^* d_1 + (1-\lambda^*) d_2](\bar{x} - f) < 1$ and $d_i(\bar{x} - f) < 1$ for $i = 2, \ldots, t$. Since P is S-free, $d_1(\bar{x} - f) > 1$. Thus there exists $\bar{\lambda} > \lambda^*$ such that $[\bar{\lambda} d_1 + (1 - \bar{\lambda}) d_2](\bar{x} - f) = 1$. Note that, since $P(\bar{\lambda}) \cap \mathrm{conv}(S)$ is bounded, there is a finite number of points of S in the interior of $P(\bar{\lambda})$. So we may choose $\bar{x}$ such that $\bar{\lambda}$ is maximum. Thus $P(\bar{\lambda})$ is S-free and $\bar{x}$ is in the relative interior of the facet of $P(\bar{\lambda})$ defined by $[\bar{\lambda} d_1 + (1 - \bar{\lambda}) d_2](x - f) \leq 1$.

Note that, for $i = 2, \ldots, t$, if $d_i(x - f) \leq 1$ defines a facet of P with a point of S in its relative interior, then it also defines a facet of $P(\bar{\lambda})$ with a point of S in its relative interior, because $P \subset P(\bar{\lambda})$. Thus repeating the above construction at most t times, we obtain a set B satisfying the lemma. □

Remark 48. *Let C and K be as in Lemma 47. Given any maximal S-free convex set $B = \{x \in \mathbb{R}^n \,|\, a_i(x - f) \leq 1,\ i = 1, \ldots, k\}$ containing C, then $a_1, \ldots, a_k \in K^*$. If $\mathrm{rec}(C)$ is not full dimensional, then the origin is not an extreme point of K^*. Since all extreme points of K^* are contained in $\{0\} \cup \hat{K}$, in this case $\mathbf{cl}(\mathrm{conv}(\hat{K})) = K^*$. Therefore, when $\mathrm{rec}(C)$ is not full dimensional, every maximal S-free convex set containing C satisfies the statement of Lemma 47.*

Proof of Theorem 44.

We first show that any valid function is dominated by a function of the form ψ_B, for some maximal S-free convex set B containing f in its interior.

Let ψ be a valid function. By Lemma 45, we may assume that ψ is sublinear. Let $K = \{r \in \mathbb{R}^n \mid \psi(r) \leq 1\}$, and let $\hat{K}$ be defined as in (2.3). Note that $K = B_\psi - f$. Thus, by Remark 46, $\sum_{r \in \mathbb{R}^n} \rho_K(r) s_r \geq 1$ is valid for $R_{f,S}$. Since ψ is sublinear, it follows from Theorem 15 that $\rho_K(r) \leq \psi(r)$ for every $r \in \mathbb{R}^n$.

By Lemma 47, there exists a maximal S-free convex set $B = \{x \in \mathbb{R}^n \mid a_i(x - f) \leq 1,\ i = 1, \ldots, k\}$ such that $a_i \in \mathbf{cl}(\mathrm{conv}(\hat{K}))$ for $i = 1, \ldots, k$.

Then

$$\psi(r) \geq \rho_K(r) = \sup_{y \in \hat{K}} yr = \max_{y \in \mathbf{cl}(\mathrm{conv}(\hat{K}))} yr \geq \max_{i=1,\ldots,k} a_i r = \psi_B(r).$$

This shows that ψ_B dominates ψ for all $r \in \mathbb{R}^n$.

To complete the proof of the theorem, we need to show that, given a maximal S-free convex set B, the function ψ_B is minimal. Consider any valid function ψ dominating ψ_B. Then by definition of B_ψ and Remark 41, $B_\psi \supseteq B$ and B_ψ is S-free. By maximality of B, $B = B_\psi$. By Theorem 15 and Remark 37, $\psi_B(r) \leq \psi(r)$ for all $r \in \mathbb{R}^n$, proving $\psi = \psi_B$.

□

Example 49. *We illustrate the ideas behind the proof in the following two-dimensional example. Consider $f = (\frac{1}{4}, \frac{1}{2})$ and $S = \{(x_1, x_2) \mid x_1 \geq 0)\}$. See Figure 3.5. Then the function $\psi(r) = \max\{4r_1 + 8r_2, 4r_1 - 8r_2\}$ is a valid linear inequality for $R_{f,S}$. The corresponding B_ψ is $\{(x_1, x_2) \mid 4(x_1 - \frac{1}{4}) + 8(x_2 - \frac{1}{2}) \leq 1, 4(x_1 - \frac{1}{4}) - 8(x_2 - \frac{1}{2}) \leq 1\}$. Note that B_ψ is not a maximal S-free convex set and it corresponds to C in Lemma 47. Following the procedure outlined in the proof, we obtain the maximal S-free convex set $B = \{(x_1, x_2) \mid 4(x_1 - \frac{1}{4}) + 4(x_2 - \frac{1}{2}) \leq 1, 4(x_1 - \frac{1}{4}) - 4(x_2 - \frac{1}{2}) \leq 1\}$. Then, $\psi_B(r) = \max\{4r_1 + 4r_2, 4r_1 - 4r_2\}$ and ψ_B dominates ψ.*

Remark 50. *Note that ψ is nonnegative if and only if $\mathrm{rec}(B_\psi)$ is not full-dimensional. It follows from Remark 48 that, for every maximal S-free convex set B containing B_ψ, we have $\psi_B(r) \leq \psi(r)$ for every $r \in \mathbb{R}^n$ when ψ is nonnegative.*

As discussed in Section 3.1, Borozan-Cornuéjols [25] study a model similar to (3.9) when $S = \mathbb{Z}^n$ and the vectors s are elements of $\mathbb{R}^{\mathbb{Q}^n}$. In this case, it is easy

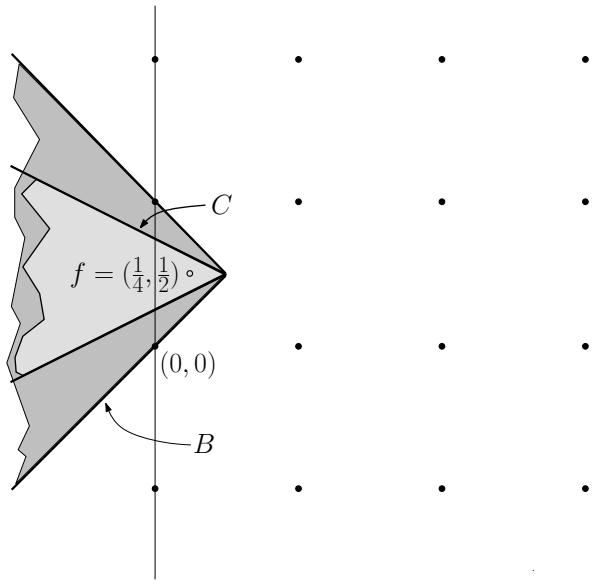


Figure 3.5: Illustration for Example 49

to show that, for every valid inequality $\sum_{r \in \mathbb{Q}^n} \psi(r) s_r \geq 1$, the function $\psi : \mathbb{Q}^n \to \mathbb{R}$ is nonnegative. Remark 50 explains why in this context it is much easier to prove that minimal inequalities arise from maximal lattice-free convex sets.

Chapter 4

Lifting Integer Constrained Non-basic Variables

In the previous Chapter, we studied two models for integer programming and showed how to derive generic functions or "formulas" for the purpose of deriving cutting planes. As shown in the previous Chapter, these functions have a direct correspondence with maximal convex sets with no feasible integer point in their interior. A crucial aspect of these models was the relaxation of the integrality constraints on non-basic variables. In this chapter, we try to deal with this issue.

During their investigation of the corner polyhedron, Gomory and Johnson took an approach where they studied valid functions for the infinite group problem. Recall that the infinite group problem (2.11) is an abstraction of the pure integer corner polyhedron problem (which means all the non-basic variables are integer constrained). They showed in a series of papers [44, 45, 48] that one could use valid inequalities for pure integer corner polyhedra to derive valid inequalities for mixed integer corner polyhedra also. The coefficients for the continuous non-basic variables were obtained as a well-defined limit of the coefficients for the integer constrained non-basic variables.

In recent years, a converse approach has been taken to the mixed integer problem. This was partly fueled by the understanding gained about valid inequalities for the models where the integrality constraints on the non-basic variables was dropped. The results discussed in Chapter 3 give very nice geometric characterizations of these valid inequalities. Given these results, a new line of research has been to use the minimal valid functions described in Chapter 3 to find coefficients for integer constrained non-basic variables. We now formally describe the setup.

We consider the following model, which puts integrality constraints on some of the non-basic variables in the model (2.12) (this is simply a reformulation of the corner polyhedron formulation).

$$\begin{array}{rcl} x & = & f + \sum_{i=1}^{k} r^i s_i \\ x & \in & \mathbb{Z}^m \\ s_i & \geq & 0 \quad \text{for all } i \\ s_i & \in & \mathbb{Z} \quad \text{for all } i \in K \subseteq \{1, \ldots, k\} \end{array} \tag{4.1}$$

A classical family of cutting planes for (4.1) is that of Gomory mixed integer cuts [55]. For a given row (say indexed by i) of the tableau, the Gomory mixed integer cut is of the form $\sum_{j \notin K} \psi(r^j{}_i) s_j + \sum_{j \in K} \pi(r^j{}_i) s_j \geq 1$ where ψ and π are functions given by simple formulas. A nice feature of the Gomory mixed integer cut is that, for fixed f_i, the same functions ψ, π are used for any possible choice of the a_{ij}'s in (4.1). It is well known that the Gomory mixed integer cuts are valid for (4.1). Therefore, Gomory Mixed-Integer functions are a concrete example of "formulas" or valid functions for mixed integer corner polyhedra, analogous to the concept of valid functions developed in Chapter 3.

In the same spirit as the definitions in (3.2) and (3.10) in Chapter 3, we are interested in pairs (ψ, π) of functions such that the inequality $\sum_{j \notin K} \psi(r^j) x_j + \sum_{j \in K} \pi(r^j) x_j \geq 1$ is valid for (4.1) for any possible choice of the nonbasic coefficients a_{ij}. As in the previous models from Chapter 3, we will search for functions (ψ, π) that are pointwise minimal. While a general characterization of minimal valid functions seems hopeless (see for example [20]), when $K = \emptyset$ the minimal valid functions ψ are well understood in terms of maximal lattice-free convex sets, as already described in Chapter 3. Starting from such a minimal valid function ψ, an interesting question is how to generate a function π such that (ψ, π) is valid and minimal. Recent papers [28], [37] study when such a function π is unique. Here we prove a theorem that generalizes and unifies results from these two papers. This work was done by the author in collaboration with Manoel Campelo, Michele Conforti, Gerard Cornuejols and Giacomo Zambelli and the results are also presented in [19].

In order to formalize the concept of valid function (ψ, π), we introduce the following infinite model, along the same lines as (2.13) and (3.9). In the setting below, we also allow further linear constraints on the basic variables, like in the case of (3.9) in Section 3.2. As before, let S be the set of integral points in some rational polyhedron in $\mathbb{R}^n$ such that $\dim(S) = n$ (for example, S could be the set of nonnegative integer points). Let $f \in \mathbb{R}^n \setminus S$. Consider the following semi-infinite

relaxation of (4.1).

$$\begin{aligned} x = \; & f + \sum_{r \in \mathbb{R}^n} r s_r + \sum_{r \in \mathbb{R}^n} r y_r, \\ & x \in S, \\ & s_r \in \mathbb{R}_+, \, \forall r \in \mathbb{R}^n, \\ & y_r \in \mathbb{Z}_+, \, \forall r \in \mathbb{R}^n, \\ & s, y \text{ have finite support} \end{aligned} \tag{4.2}$$

where the nonbasic continuous variables have been renamed s and the nonbasic integer variables have been renamed y. Given two functions $\psi, \pi : \mathbb{R}^n \to \mathbb{R}$, (ψ, π) is said to be *valid* for (4.2) if the inequality $\sum_{r \in \mathbb{R}^n} \psi(r) s_r + \sum_{r \in \mathbb{R}^n} \pi(r) y_r \geq 1$ holds for every (x, s, y) satisfying (4.2).

Given a valid function ψ for (3.9), a function π is a *lifting* of ψ if (ψ, π) is valid for (4.2). One is interested only in (pointwise) *minimal valid functions*, for the same reasons as discussed in detail in Chapter 3. If ψ is a minimal valid function for (3.9) and π is a lifting of ψ such that (ψ, π) is a minimal valid function for (4.2) then we say that π is a *minimal lifting* of ψ.

While minimal valid functions for (3.9) have a simple characterization (Theorem 44), minimal valid functions for (4.2) are not well understood. For example, if (ψ, π) is a minimal valid function, it does not imply that ψ is a minimal valid function for (3.9). Moreover, minimal valid functions for (3.9) are piecewise linear because they are of the form ψ_B for some maximal S-free convex set B. However, there are valid functions (ψ, π) for (4.2) such that π is not piecewise linear. This result is presented in Chapter 7. A general idea to derive minimal valid functions for (4.2) is to start from some minimal valid function ψ for (3.9), and construct a minimal lifting π of ψ. While there is no general technique to compute such minimal lifting π, it is known that there exists a region R_ψ, containing the origin in its interior, where ψ coincides with π for any minimal lifting π. This latter fact was observed by Dey and Wolsey [37] for the case of $S = \mathbb{Z}^2$ and by Conforti, Cornuéjols and Zambelli [28] for the general case. Furthermore, it is remarked in [28] and [37] that, if π is a minimal lifting of ψ, then $\pi(r) = \pi(r')$ for every $r, r' \in \mathbb{R}^n$ such that $r - r' \in \mathbb{Z}^n \cap \operatorname{lin}(\operatorname{conv}(S))$. Therefore the coefficients of any minimal lifting π are uniquely determined in the region $R_\psi + (\mathbb{Z}^n \cap \operatorname{lin}(\operatorname{conv}(S)))$. In particular, whenever translating R_ψ by integer vectors in $\operatorname{lin}(\operatorname{conv}(S))$ covers $\mathbb{R}^n$, ψ has a unique minimal lifting. In this Chapter, we give a precise description of the region R_ψ.

We first observe the following corollary which follows from the proof of Theorem 44.

Corollary 51. *Let ψ be a minimal valid function for* (3.9)*. Then the set*

$$B_\psi := \{x \in \mathbb{R}^n \mid \psi(x - f) \leq 1\}$$

is a maximal S-free convex set containing f in its interior, and $\psi = \psi_{B_\psi}$.

We are now ready to state the main result of the paper. Given a minimal valid function ψ for (3.9), by Corollary 51 B_ψ is a maximal S-free convex set containing f in its interior, thus it can be uniquely written as $B_\psi = \{x \in \mathbb{R}^n \mid a_i(x-f) \leq 1,\, i \in I\}$. For every $r \in \mathbb{R}^n$, let $I(r) = \{i \in I \mid \psi(r) = a_i r\}$. Given $x \in S$, let

$$R(x) := \{r \in \mathbb{R}^n \mid I(r) \supseteq I(x - f) \text{ and } I(x - f - r) \supseteq I(x - f)\}.$$

We define

$$R_\psi := \bigcup_{x \in S \cap B_\psi} R(x).$$

Theorem 52. *Let ψ be a minimal valid function for* (3.9)*. If π is a minimal lifting of ψ, then $\pi(r) = \psi(r)$ for every $r \in R_\psi$.*
Conversely, for every $\bar{r} \notin R_\psi$, there exists a lifting π of ψ such that $\pi(\bar{r}) < \psi(\bar{r})$.

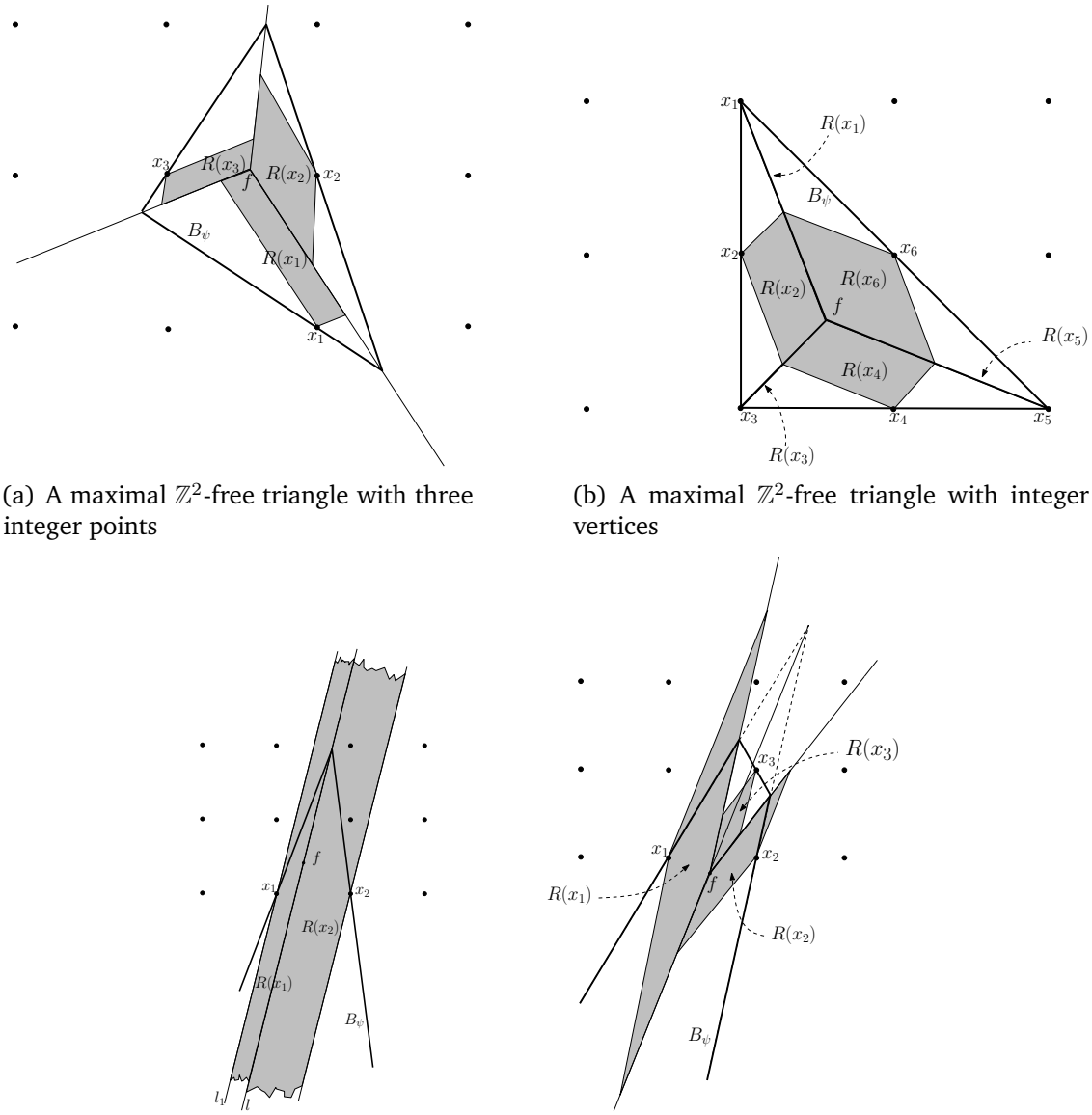
Figure 4.1 illustrates the region R_ψ for several examples. We conclude the introduction presenting a different characterization of the regions $R(x)$.

Proposition 53. *Let ψ be a minimal valid function for* (3.9)*, and let $x \in S$. Then $R(x) = \{r \in \mathbb{R}^n \mid \psi(r) + \psi(x - f - r) = \psi(x - f)\}$.*

Proof. We can uniquely write $B_\psi = \{x \in \mathbb{R}^n \mid a_i(x-f) \leq 1,\, i \in I\}$. Let $h \in I(x-f)$. Then

$$\psi(x-f) = a_h(x-f) = a_h r + a_h(x-f-r) \leq \max_{i \in I} a_i r + \max_{i \in I} a_i(x-f-r) = \psi(r) + \psi(x-f-r).$$

In the above expression, equality holds if and only if $h \in I(r)$ and $h \in I(x - f - r)$. $\square$



(a) A maximal $\mathbb{Z}^2$-free triangle with three integer points

(b) A maximal $\mathbb{Z}^2$-free triangle with integer vertices

(c) A wedge

(d) A truncated wedge

Figure 4.1: Regions $R(x)$ for some maximal S-free convex sets in the plane. The thick dark line indicates the boundary of B_ψ. For a particular x, the dark gray regions denote $R(x)$. The jagged lines in a region indicate that it extends to infinity. For example, in Figure 4.1(c), $R(x_1)$ is the strip between lines l_1 and l. Figure 4.1(b) shows an example where $R(x)$ is full-dimensional for x_2, x_4, x_6, but is not full-dimensional for x_1, x_3, x_5.

4.1 Minimum lifting coefficient of a single variable

Given $r^* \in \mathbb{R}^n$, we consider the set of solutions to

$$\begin{aligned} x = \quad & f + \sum_{r \in \mathbb{R}^n} r s_r + r^* y_{r^*} \\ & x \in S \\ & s \geq 0 \\ & y_{r^*} \geq 0,\ y_{r^*} \in \mathbb{Z} \\ & s \text{ has finite support.} \end{aligned} \tag{4.3}$$

Given a minimal valid function ψ for (3.9) and scalar λ, we say that the inequality $\sum_{r \in \mathbb{R}^n} \psi(r) s_r + \lambda y_{r^*} \geq 1$ is valid for (4.3) if it holds for every (x, s, y_{r^*}) satisfying (4.3). We denote by $\psi^*(r^*)$ the minimum value of λ for which $\sum_{r \in \mathbb{R}^n} \psi(r) s_r + \lambda y_{r^*} \geq 1$ is valid for (4.3).

We observe that, for *any* lifting π of ψ, we have

$$\psi^*(r^*) \leq \pi(r^*).$$

Indeed, $\sum_{r \in \mathbb{R}^n} \psi(r) s_r + \pi(r^*) y_{r^*} \geq 1$ is valid for (4.3), since, for any $(\bar{s}, \bar{y}_{r^*})$ satisfying (4.3), the vector $(\bar{s}, \bar{y})$, defined by $\bar{y}_r = 0$ for all $r \in \mathbb{R}^n \setminus \{r^*\}$, satisfies (4.2).

Moreover, the following fact was shown in [28]. We include the proof here for completeness.

Lemma 54. *If ψ is a minimal valid function for* (3.9) *and π is a minimal lifting of ψ, then $\pi \leq \psi$.*

Proof. Given a lifting π of ψ, consider the function $\pi'(r) = \min\{\pi(r), \psi(r)\}$. We claim that π' is a valid lifting of ψ. Consider any (s, y) that satisfies (4.2). Construct a new solution $(\bar{s}, \bar{y})$ in the following way. $(\bar{s})_r = s_r, (\bar{y})_r = y_r$ for all r such that $\pi(r) \leq \psi(r)$ and $(\bar{s})_r = s_r + y_r, (\bar{y})_r = 0$ if $\pi(r) > \psi(r)$. It can be verified that $(\bar{s}, \bar{y})$ is a valid solution to (4.2). Therefore, $\sum_{r \in \mathbb{R}^n} \psi(r)(\bar{s})_r + \sum_{r \in \mathbb{R}^n} \pi(r)(\bar{y})_r \geq 1$. Moreover, $\sum_{r \in \mathbb{R}^n} \psi(r) s_r + \sum_{r \in \mathbb{R}^n} \pi'(r) y_r = \sum_{r \in \mathbb{R}^n} \psi(r)(\bar{s})_r + \sum_{r \in \mathbb{R}^n} \pi'(r)(\bar{y})_r \geq 1$. Since the choice of (s, y) was arbitrary, ψ, π' is a valid function. □

So we have the following relation for every minimal lifting π of ψ :

$$\psi^*(r) \leq \pi(r) \leq \psi(r) \quad \text{for all } r \in R^n.$$

In general ψ^* is not a lifting of ψ, but if it is, then the above relation implies that it is the unique minimal lifting of ψ.

Remark 55. *For any $r \in \mathbb{R}^n$ such that $\psi^*(r) = \psi(r)$, we have $\pi(r) = \psi(r)$ for every minimal lifting π of ψ. Conversely, if $\psi^*(r^*) < \psi(r^*)$ for some $r^* \in \mathbb{R}^n$, then there exists some lifting π of ψ such that $\pi(r^*) < \psi(r^*)$.*

Proof. The first part follows from $\psi^* \leq \pi \leq \psi$. For the second part, given $r^* \in \mathbb{R}^n$ such that $\psi^*(r^*) < \psi(r^*)$, we can define π by $\pi(r^*) = \psi^*(r^*)$ and $\pi(r) = \psi(r)$ for all $r \in \mathbb{R}^n$, $r \neq r^*$. Since ψ is valid for (3.9), it follows by the definition of $\psi^*(r^*)$ that π is a lifting of ψ. □

By the above remark, in order to prove Theorem 52 we need to show that $R_\psi = \{r \in \mathbb{R}^n \mid \psi(r) = \psi^*(r)\}$. We will need the following results.

Remark 56. *The proof of Theorem 42 implies the following. Given a maximal S-free convex set B, there exists $\delta > 0$ such that there is no point of $S \setminus B$ at distance less than δ from B.*

Let $r^* \in \mathbb{R}^n$. Given a maximal S-free convex set $B = \{x \in \mathbb{R}^n \mid a_i(x-f) \leq 1,\ i \in I\}$, for any $\lambda \in \mathbb{R}$, we define the set $B(\lambda) \subset \mathbb{R}^{n+1}$ as follows

$$B(\lambda) = \{\begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1} \mid a_i(x-f) + (\lambda - a_i r^*)x_{n+1} \leq 1,\ i \in I\}. \tag{4.4}$$

Theorem 57. *[28] Let $r^* \in \mathbb{R}^n$. Given a maximal S-free convex set B, let $\psi = \psi_B$. Given $\lambda \in \mathbb{R}$, the inequality $\sum_{r \in \mathbb{R}^n} \psi(r)s_r + \lambda y_{r^*} \geq 1$ is valid for* (4.3) *if and only if $B(\lambda)$ is $(S \times \mathbb{Z}_+)$-free.*

Remark 58. *Let $r^* \in \mathbb{R}^n$, and let B be a maximal S-free convex set. For every λ such that $B(\lambda)$ is $(S \times \mathbb{Z}_+)$-free, $B(\lambda)$ is maximal $(S \times \mathbb{Z}_+)$-free.*

Proof. Since B is a maximal S-free convex set, then by Theorem 42 each facet of B contains a point $\bar{x}$ of S in its relative interior. Therefore the corresponding facet of $B(\lambda)$ contains the point $\binom{\bar{x}}{0}$ in its relative interior. If $B(\lambda)$ is $(S \times \mathbb{Z}_+)$-free, by Theorem 42 it is a maximal $(S \times \mathbb{Z}_+)$-free convex set. □

4.2 Characterizing the region R_ψ

Next we state and prove a theorem that characterizes when $\psi^*(r^*) = \psi(r^*)$. The main result of this paper, Theorem 52, will then follow easily.

Theorem 59. *Given a maximal S-free convex set B, let $\psi = \psi_B$. Given $r^* \in \mathbb{R}^n$, the following are equivalent:*
(i) $\psi^*(r^*) = \psi(r^*)$.
(ii)There exists a point $\bar{x} \in S$ such that $\binom{\bar{x}}{1} \in B(\psi(r^*))$.

Proof. Let $\lambda^* = \psi(r^*)$. Note that the inequality $\sum_{r\in\mathbb{R}^n} \psi(r)s_r + \lambda^* y_{r^*} \geq 1$ is valid for (4.3). Thus, it follows from Theorem 57 that $B(\lambda^*)$ is $S \times \mathbb{Z}_+$-free .

We first show that (ii) implies (i). Assume there exists $\bar{x} \in S$ such that $\binom{\bar{x}}{1} \in B(\lambda^*)$. Then, for every $\epsilon > 0$, $\binom{\bar{x}}{1}$ is in the interior of $B(\lambda^* - \epsilon)$, because $a_i(\bar{x} - f) + (\lambda - \varepsilon - a_i r^*) \leq 1 - \varepsilon < 1$ for all $i \in I$. Theorem 57 then implies that $\psi^*(r^*) = \lambda^*$.

Next we show that (i) implies (ii). Assume that $\psi^*(r^*) = \psi(r^*) = \lambda^*$. We recall that $\lambda^* = \max_{i\in I} a_i r^*$.

Note that, if $a_i r^* = \lambda^*$ for all $i \in I$, then $B(\lambda^*) = B \times \mathbb{R}$, so given any point $\bar{x}$ in $B \cap S$, $\binom{\bar{x}}{1}$ is in $B(\lambda^*)$. Thus we assume that there exists an index h such that $a_h r^* < \lambda^*$.

By Remark 58, $B(\lambda^*)$ is maximal $(S \times \mathbb{Z}_+)$-free. Theorem 42 implies the following,
a) $\mathrm{rec}(B \cap \mathrm{conv}(S)) \subseteq \mathrm{lin}(B)$,
b) $\mathrm{rec}(B(\lambda^*) \cap \mathrm{conv}(S \times \mathbb{Z}_+)) \subseteq \mathrm{lin}(B(\lambda^*))$.

Claim 14. $\mathrm{rec}(B(\lambda^*) \cap \mathrm{conv}(S \times \mathbb{Z}_+)) = \mathrm{rec}(B \cap \mathrm{conv}(S)) \times \{0\}$.

Let $\binom{\bar{r}}{\bar{r}_{n+1}} \in \mathrm{rec}(B(\lambda^*) \cap \mathrm{conv}(S \times \mathbb{Z}_+))$. Note that $\mathrm{rec}(\mathrm{conv}(S \times \mathbb{Z}_+)) = \mathrm{rec}(\mathrm{conv}(S)) \times \mathbb{Z}_+$, thus $\bar{r} \in \mathrm{rec}(\mathrm{conv}(S))$ and $\bar{r}_{n+1} \geq 0$. We only need to show that $\bar{r}_{n+1} = 0$.

By b), $\binom{\bar{r}}{\bar{r}_{n+1}}$ satisfies

$$a_i \bar{r} + (\lambda^* - a_i r^*)\bar{r}_{n+1} = 0, \quad i \in I. \tag{4.5}$$

Since $\lambda^* - a_i r^* \geq 0$ and $\bar{r}_{n+1} \geq 0$,

$$a_i \bar{r} \leq 0, \quad i \in I,$$

therefore $\bar{r} \in \mathrm{rec}(B)$. Thus $\bar{r} \in \mathrm{rec}(B \cap \mathrm{conv}(S))$ which, by a), is contained in $\mathrm{lin}(B)$. This implies

$$a_i \bar{r} = 0, \quad i \in I.$$

It follows from the above and from (4.5) that $(\lambda^* - a_i r^*)\bar{r}_{n+1} = 0$ for $i \in I$. Since $\lambda^* - a_h r^* > 0$ for some index h, it follows that $\bar{r}_{n+1} = 0$. This concludes the proof of Claim 14.

Claim 15. *There exists* $\bar{\varepsilon} > 0$ *such that* $\mathrm{rec}(B(\lambda^* - \varepsilon) \cap \mathrm{conv}(S \times \mathbb{Z}_+)) = \mathrm{rec}(B \cap \mathrm{conv}(S)) \times \{0\}$ *for every* $\varepsilon \in [0, \bar{\varepsilon}]$.

Since $\mathrm{conv}(S)$ is a rational polyhedron, $S = \{x \in \mathbb{R}^n \mid Cx \leq d\}$ for some rational matrix (C, d). By Claim 14, there is no vector $\binom{r}{1}$ in $\mathrm{rec}(B(\lambda^*) \cap \mathrm{conv}(S \times \mathbb{Z}_+))$. Thus the system

$$\begin{array}{rcl} a_i r + (\lambda^* - a_i r^*) & \leq & 0, \quad i \in I \\ Cr & \leq & 0 \end{array}$$

is infeasible. By Farkas Lemma, there exist scalars $\mu_i \geq 0$, $i \in I$ and a nonnegative vector γ such that

$$\begin{array}{rcl} \sum_{i \in I} \mu_i a_i + \gamma C & = & 0 \\ \lambda^* (\sum_{i \in I} \mu_i) - (\sum_{i \in I} \mu_i a_i) r^* & > & 0. \end{array}$$

This implies that there exists some $\bar{\varepsilon} > 0$ such that for all $\varepsilon \leq \bar{\varepsilon}$,

$$\begin{array}{rcl} \sum_{i \in I} \mu_i a_i + \gamma C & = & 0 \\ (\lambda^* - \varepsilon)(\sum_{i \in I} \mu_i) - (\sum_{i \in I} \mu_i a_i) r^* & > & 0, \end{array}$$

thus the system

$$\begin{array}{rcl} a_i r + (\lambda^* - \varepsilon - a_i r^*) & \leq & 0, \quad i \in I \\ Cr & \leq & 0 \end{array}$$

is infeasible. This implies that $\mathrm{rec}(B(\lambda^* - \varepsilon) \cap \mathrm{conv}(S \times \mathbb{Z}_+)) = \mathrm{rec}(B \cap \mathrm{conv}(S)) \times \{0\}$.

Claim 16. $B(\lambda^*)$ *contains a point* $\binom{\bar{x}}{\bar{x}_{n+1}} \in S \times \mathbb{Z}_+$ *such that* $\bar{x}_{n+1} > 0$.

By Claim 15, there exists $\bar{\varepsilon}$ such that, for every $\varepsilon \in [0, \bar{\varepsilon}]$, $\mathrm{rec}(B(\lambda^* - \varepsilon) \cap \mathrm{conv}(S \times \mathbb{Z}_+)) = \mathrm{rec}(B \cap \mathrm{conv}(S)) \times \{0\}$. This implies that there exists a scalar M such that, for every $\varepsilon \in [0, \bar{\varepsilon}]$ and every point $\binom{\bar{x}}{\bar{x}_{n+1}} \in B(\lambda^* - \varepsilon) \cap (S \times \mathbb{Z}_+)$, it follows $\bar{x}_{n+1} \leq M$.

Remark 56 and Remark 58 imply that there exists $\delta > 0$ such that, for every $\binom{\bar{x}}{\bar{x}_{n+1}} \in (S \times \mathbb{Z}_+) \setminus B(\lambda^*)$, there exists $h \in I$ such that $a_h(\bar{x} - f) + (\lambda^* - a_h r^*)\bar{x}_{n+1} \geq 1 + \delta$. Choose $\varepsilon > 0$ such that $\varepsilon \leq \bar{\varepsilon}$ and $\varepsilon M \leq \delta$.

Since $\psi^*(r^*) = \lambda^*$, by Theorem 57, $B(\lambda^* - \varepsilon)$ has a point $\binom{\bar{x}}{\bar{x}_{n+1}} \in S \times \mathbb{Z}_+$ in its interior. Thus $a_i(\bar{x} - f) + (\lambda^* - \varepsilon - a_i r^*)\bar{x}_{n+1} < 1$, $i \in I$.

We show that $\binom{\bar{x}}{\bar{x}_{n+1}}$ is also in $B(\lambda^*)$. Suppose not. Then, by our choice of δ, there exists $h \in I$ such that $a_h(\bar{x} - f) + (\lambda^* - a_h r^*)\bar{x}_{n+1} \geq 1 + \delta$. By our choice of M and ε,

$$1+\delta \leq a_h(\bar{x}-f)+(\lambda^*-a_h r^*)\bar{x}_{n+1} \leq a_h(\bar{x}-f)+(\lambda^*-\varepsilon-a_h r^*)\bar{x}_{n+1}+\varepsilon M < 1+\varepsilon M \leq 1+\delta,$$

a contradiction.

Hence $\binom{\bar{x}}{\bar{x}_{n+1}}$ is in $B(\lambda^*)$. Since B is S-free and $B(\lambda^* - \varepsilon) \cap (\mathbb{R}^n \times \{0\}) = B \times \{0\}$, it follows that $B(\lambda^* - \varepsilon)$ does not contain any point of $S \times \{0\}$ in its interior. Thus $\bar{x}_{n+1} > 0$. This concludes the proof of Claim 16.

By the previous claim, $B(\lambda^*)$ contains a point $\binom{\bar{x}}{\bar{x}_{n+1}} \in S \times \mathbb{Z}_+$ such that $\bar{x}_{n+1} > 0$. Note that $B(\lambda^*)$ contains $\binom{\bar{x}}{1}$, since

$$a_i(\bar{x} - f) + (\lambda^* - a_i r^*) \leq a_i(\bar{x} - f) + (\lambda^* - a_i r^*)\bar{x}_{n+1} \leq 1, \quad i \in I$$

since $\lambda^* - a_i r^* \geq 0$, $i \in I$. □

Corollary 60. *Let ψ be a minimal valid function for* (3.9). *Then $\psi^*(r^*) = \psi(r^*)$ if and only if there exists $\bar{x} \in S$ such that*

$$\psi(r^*) + \psi(\bar{x} - f - r^*) = 1. \tag{4.6}$$

Proof. We first show that, if there exist $\bar{x} \in S$ satisfying (4.6), then $\psi^*(r^*) = \psi(r^*)$. Indeed, since $\sum_{r \in \mathbb{R}^n} \psi(r) s_r + \psi^*(r^*) y_{r^*} \geq 1$ is valid for (4.3),

$$1 \leq \psi^*(r^*) + \psi(\bar{x} - f - r^*) \leq \psi(r^*) + \psi(\bar{x} - f - r^*) = 1.$$

We show the converse. Since ψ is a valid function for (3.9), $\psi(\bar{x} - f - r^*) + \psi(r^*) \geq 1$. Since ψ is a minimal valid function for (3.9), by Corollary 51 there exists

a maximal S-free convex set B such that $\psi = \psi_B$. Let $B_\psi = \{x \in \mathbb{R}^n \mid a_i(x - f) \leq 1,\, i \in I\}$.

Assume $\psi^*(r^*) = \psi(r^*)$. By Theorem 59, there exists a point $\bar{x} \in S$ such that $\binom{\bar{x}}{1} \in B(\psi(r^*))$. Therefore

$$a_i(\bar{x} - f) + \psi(r^*) - a_i r^* \leq 1, \quad i \in I.$$

Thus

$$\max_{i \in I} a_i(\bar{x} - f - r^*) \leq 1 - \psi(r^*),$$

which implies $\psi(\bar{x} - f - r^*) + \psi(r^*) \leq 1$. Hence $\psi(\bar{x} - f - r^*) + \psi(r^*) = 1$. □

Proof of Theorem 52. By Remark 55, we only need to show that $R_\psi = \{r \in \mathbb{R}^n \mid \psi(r) = \psi^*(r)\}$. For every $x \in S$, we have $\psi(x - f) = 1$ if and only if $x \in S \cap B_\psi$. Therefore, by Proposition 53, $R(x) = \{r \in \mathbb{R}^n \mid \psi(r) + \psi(x - f - r) = \psi(x - f) = 1\}$ if and only if $x \in S \cap B_\psi$. The latter fact and Corollary 60 imply that a vector $r \in \mathbb{R}^n$ satisfies $\psi^*(r) = \psi(r)$ if and only if $r \in R(x)$ for some $x \in S \cap B_\psi$. The statement now follows from the definition of R_ψ. □

4.3 Conclusion

In this Chapter, we give an exact characterization of the region where a minimal valid inequality ψ and any minimal lifting π of ψ coincide. This was exhibited in Theorem 52, which generalizes results from [28] and [37] about liftings of minimal valid inequalities.

As already mentioned in the introduction, the following theorem was proved in [28].

Theorem 61. *Let ψ be a minimal valid function for* (3.9). *If $R_\psi + (\mathbb{Z}^n \cap \operatorname{lin}(\operatorname{conv}(S)))$ covers all of $\mathbb{R}^n$, then there exists a unique minimal lifting π of ψ.*

We conjecture that the converse also holds.

Conjecture 62. *Let ψ be a minimal valid function for* (3.9). *There exists a unique minimal lifting π of ψ if and only if $R_\psi + (\mathbb{Z}^n \cap \operatorname{lin}(\operatorname{conv}(S)))$ covers all of $\mathbb{R}^n$.*

Chapter 5

Relative Strength of Split, Triangle and Quadrilateral Cuts

The purpose of this chapter is to establish theoretical guarantees for the superiority of cutting planes from multiple constraints over the single-constraint cutting planes like Split Cuts. In particular, we study cutting planes derived from two constraints of the optimal tableau. The natural setting for studying these cuts is to study the corner polyhedron with two constraints. In fact, we will study the relaxation (2.12), with two equality constraints. Andersen et. al studied this two constraint system in [8] and the connection with maximal lattice-free sets was observed as described in Chapter 3. We focus our attention on a slightly different aspect of this system in this chapter. The maximal lattice-free sets which give valid inequalities for the two constraint system are well-known in the literature [37, 34]. Here we compare the strength of these different families of inequalities corresponding to the different lattice-free sets. The results in this chapter were obtained by the author in collaboration with Pierre Bonami, Gerard Cornuejols and Francois Margot in [18]. Therefore, we will be studying the following system

$$\begin{array}{rcl} x & = & f + \sum_{j=1}^{k} r^j s_j \\ x & \in & \mathbb{Z}^2 \\ s & \in & \mathbb{R}^k_+. \end{array} \tag{5.1}$$

Recall that we denote the convex hull of all solutions to (5.1) by $\mathrm{conv}(R_f(r^1, \ldots, r^k))$. We assume in this Chapter that all the data is rational. Meyer's theorem 23 implies that $\mathrm{conv}(R_f(r^1, \ldots, r^k))$ is a polyhedron. Andersen et al. [8] and Cornuejols and

Margot [34] study the facets of $\mathrm{conv}(R_f(r^1, \ldots, r^k))$ and it is shown in these papers that the nontrivial facets come from the restrictions of minimal valid inequalities related to maximal lattice-free convex sets in $\mathbb{R}^2$. These maximal lattice-free convex sets are splits, triangles, and quadrilaterals as proved in the following theorem of Lovász [52].

Theorem 63. (Lovász [52]) *In the plane, a maximal lattice-free convex set with nonempty interior is one of the following:*

(i) A split $c \leq ax_1 + bx_2 \leq c + 1$ where a and b are coprime integers and c is an integer;

(ii) A triangle with an integral point in the interior of each of its edges;

(iii) A quadrilateral containing exactly four integral points, with exactly one of them in the interior of each of its edges; Moreover, these four integral points are vertices of a parallelogram of area 1.

Following Dey and Wolsey [37], the maximal lattice-free triangles can be partitioned into three types (see Figure 5.1):

- *Type 1 triangles*: triangles with integral vertices and exactly one integral point in the relative interior of each edge;
- *Type 2 triangles*: triangles with at least one fractional vertex v, exactly one integral point in the relative interior of the two edges incident to v and at least two integral points on the third edge;
- *Type 3 triangles*: triangles with exactly three integral points on the boundary, one in the relative interior of each edge.

Figure 5.1 shows these three types of triangles as well as a maximal lattice-free quadrilateral and a split satisfying the properties of Theorem 63.

An unbounded maximal lattice-free convex set in $\mathbb{R}^2$ is called a *split*. It has two parallel edges whose direction is called the *direction* of the split. Split inequalities for (5.1) are valid inequalities that can be derived by combining the two equations in (5.1) and by using the integrality of $\pi_1 x_1 + \pi_2 x_2$, where $\pi \in \mathbb{Z}^2$ is normal to the direction of the split. Similarly, for general MILPs, the equations can be combined into a single equality from which a split inequality is derived. Split inequalities are

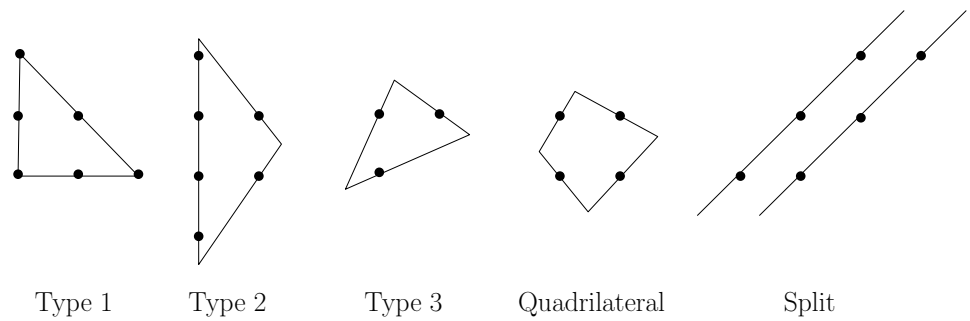


Figure 5.1: Maximal lattice-free convex sets with nonempty interior in $\mathbb{R}^2$

equivalent to Gomory mixed integer cuts [55]. Empirical evidence shows that split inequalities can be effective for strengthening the linear programming relaxation of MILPs [16, 35]. Interestingly, triangle and quadrilateral inequalities cannot be derived from a single equation [8]. They can only be derived from (5.1) without aggregating the two equations. Recent computational experiments by Espinoza [40] indicate that quadrilaterals also induce effective cutting planes in the context of solving general MILPs. In this Chapter, we consider the relative strength of split, triangle and quadrilateral inequalities from a theoretical point of view. We use an approach for measuring strength initiated by Goemans [41], based on the following definition and results.

Let $Q \subseteq \mathbb{R}^n_+ \setminus \{0\}$ be a polyhedron of the form $Q = \{x : \ a^i x \geq b_i \text{ for } i = 1, \ldots, m\}$ where $a^i \geq 0$ and $b_i \geq 0$ for $i = 1, \ldots, m$ and let $\alpha > 0$ be a scalar. We define the polyhedron αQ as $\{x : \ \alpha a^i x \geq b_i \text{ for } i = 1, \ldots, m\}$. Note that αQ contains Q when $\alpha \geq 1$. It will be convenient to define αQ to be $\mathbb{R}^n_+$ when $\alpha = +\infty$.

We need the following generalization of a theorem of Goemans [41].

Theorem 64. *Suppose $Q \subseteq \mathbb{R}^n_+ \setminus \{0\}$ is defined as above. If convex set $P \subseteq \mathbb{R}^n_+$ is a relaxation of Q (i.e. $Q \subseteq P$), then the smallest value of $\alpha \geq 1$ such that $P \subseteq \alpha Q$ is*

$$\max_{i=1,\ldots,m} \left\{ \frac{b_i}{\inf\{a^i x \ : \ x \in P\}} \ : \ b_i > 0 \right\}.$$

Here, we define $\frac{b_i}{\inf\{a^i x \ : \ x \in P\}}$ to be $+\infty$ if $\inf\{a^i x \ : \ x \in P\} = 0$.

In other words, the only directions that need to be considered to compute α are those defined by the nontrivial facets of Q.

Proof of Theorem 64. Let

$$\alpha = \max_{i=1,\ldots,m} \left\{ \frac{b_i}{\inf\{a^i x \ : \ x \in P\}} \ : \ b_i > 0 \right\}.$$

We first show that $P \subseteq \alpha Q$. This holds when $\alpha = +\infty$ by definition of αQ. Therefore we may assume $1 \leq \alpha < +\infty$. Consider any point $p \in P$. The inequalities of αQ are of the form $\alpha a^i x \geq b_i$ with $a^i \geq 0$ and $b_i \geq 0$. If $b_i = 0$, then since $p \in P \subseteq \mathbb{R}^n_+$, $a^i p \geq 0$ and hence this inequality is satisfied. If $b_i > 0$, then we know from the definition of α that

$$\frac{b_i}{\inf\{a^i x \ : \ x \in P\}} \leq \alpha.$$

This implies

$$b_i \leq \alpha \inf\{a^i x \ : \ x \in P\} \leq \alpha a^i p.$$

Therefore, p satisfies this inequality.

We next show that for any $1 \leq \alpha' < \alpha$, $P \not\subseteq \alpha' Q$. Say $\alpha = \frac{b_j}{\inf\{a^j x \ : \ x \in P\}}$ (i.e. the maximum, possibly $+\infty$, is reached for index j). Let $\delta = \frac{b_j}{\alpha'} - \frac{b_j}{\alpha}$. We have $\delta > 0$. From the definition of α we know that $\inf\{a^j x \ : \ x \in P\} = \frac{b_j}{\alpha}$. Therefore, there exists $p \in P$ such that $a^j p < \frac{b_j}{\alpha} + \delta = \frac{b_j}{\alpha'}$. So $\alpha' a^j p < b_j$ and hence $p \notin \alpha' Q$.

□

Results

Let the *split closure* $S_f(r^1, \ldots, r^k)$ be the intersection of all split inequalities, let the *triangle closure* $T_f(r^1, \ldots, r^k)$ be the intersection of all inequalities arising from maximal lattice-free triangles, and let the *quadrilateral closure* $Q_f(r^1, \ldots, r^k)$ be the intersection of all inequalities arising from maximal lattice-free quadrilaterals. In the remainder of this Chapter, to simplify notation, we refer to $\mathrm{conv}(R_f(r^1, \ldots, r^k))$, $S_f(r^1, \ldots, r^k)$, $T_f(r^1, \ldots, r^k)$ and $Q_f(r^1, \ldots, r^k)$ as R^k_f, S^k_f, T^k_f and Q^k_f respectively, whenever the rays $r^1, \ldots, r^k$ are obvious from the context.

Since all the facets of R^k_f are induced by these three families of maximal lattice-free convex sets, we have

$$R^k_f = S^k_f \cap T^k_f \cap Q^k_f.$$

It is known that the split closure is a polyhedron (Cook, Kannan and Schrijver [31]) but such a result is not known for the triangle closure and the quadrilateral closure. In this Chapter we show the following results.

Theorem 65. *$T_f^k \subseteq S_f^k$ and $Q_f^k \subseteq S_f^k$.*

This theorem may seem counter-intuitive because some split inequalities are facets of R_f^k. However, we show that any split inequality can be obtained as the limit of an infinite collection of triangle inequalities. What we will show is that, if a point is cut off by a split inequality, then it is also cut off by some triangle inequality. Consequently, the intersection of *all* triangle inequalities is contained in the split closure. The same is true of the quadrilateral closure.

Example 66. *As an illustration, consider the simple example in Figure 5.2. The split inequality is $s_1 \geq 1$ and the split closure is given by $S_f^2 = \{(s_1, s_2) \mid s_1 \geq 1, s_2 \geq 0\}$. All triangle inequalities are of the form $as_1 + bs_2 \geq 1$ with $a \geq 1$ and $b > 0$. The depicted triangle inequality is of the form $s_1 + 2s_2 \geq 1$ and by moving the corner v closer to the boundary of the split, one can get a triangle inequality of the form $s_1 + bs_2 \geq 1$ with b tending to 0, but remaining positive. Note that the split inequality can not be obtained as a positive combination of triangle inequalities, but any point cut by the split inequality is cut by one of the triangle inequalities. As a result, the triangle closure is $T_f^2 = \{(s_1, s_2) \mid s_1 + bs_2 \geq 1,$ for all $b > 0, s_1 \geq 0, s_2 \geq 0\} = S_f^2$.*

□

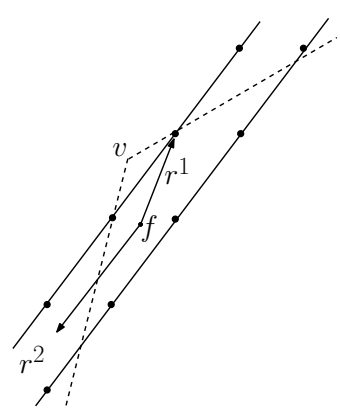


Figure 5.2: Illustration for Example 66

We further study the strength of the triangle closure and quadrilateral closure in the sense defined by Goemans (Theorem 64). We first compute the strength of a single Type 1 triangle facet as f varies in the interior of the triangle, relative to the entire split closure.

Theorem 67. *Let T be a Type 1 triangle as depicted in Figure 5.3. Let f be in its interior and assume that the set of rays $\{r^1, \ldots, r^k\}$ contains rays pointing to the three corners of T. Let $\sum_{i=1}^{k} \psi(r^i)s_i \geq 1$ be the inequality generated by T. The value*

$$\min \left\{ \sum_{i=1}^{k} \psi(r^i)s_i \ : \ s \in S_f^k \right\}$$

is a piecewise linear function of f for which some level curves are depicted in Figure 5.3. This function varies from a minimum of $\frac{1}{2}$ in the center of T to a maximum of $\frac{2}{3}$ at its corners.

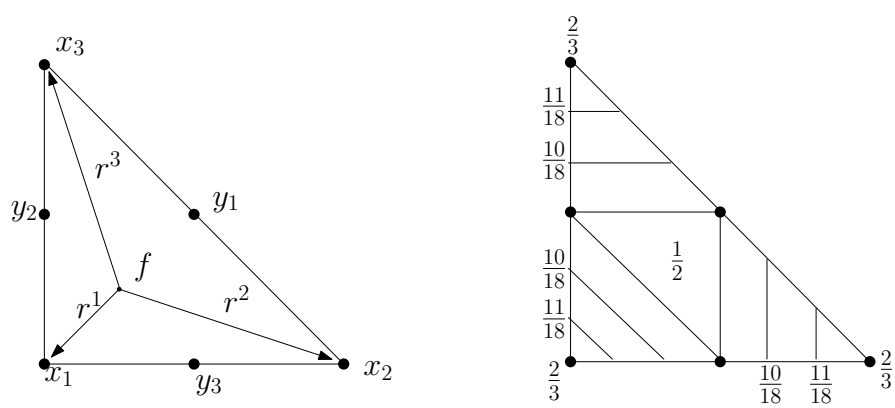


Figure 5.3: Illustration for Theorem 67

Next we show that both the triangle closure and the quadrilateral closure are good approximations of the integer hull R_f^k in the sense that

Theorem 68.

$R_f^k \subseteq T_f^k \subseteq 2R_f^k$ *and*

$R_f^k \subseteq Q_f^k \subseteq 2R_f^k$.

Finally we show that the split closure may not be a good approximation of the integer hull.

Theorem 69. *For any $\alpha > 1$, there is a choice of f, $r^1, \ldots, r^k$ such that $S_f^k \not\subseteq \alpha R_f^k$.*

5.1 Split closure vs. triangle and quadrilateral closures

In this section, we present the proof of Theorem 65.

Proof. (Theorem 65). We show that if any point $\bar{s}$ is cut off by a split inequality, then it is also cut off by some triangle inequality. This will prove the theorem.

Consider any split inequality $\sum_{i=1}^k \psi_S(r^i)s_i \geq 1$ (see Figure 5.4) and denote by L_1 and L_2 its two boundary lines. Point f lies in some parallelogram of area 1 whose vertices $y^1, y^2, y^3,$ and y^4 are lattice points on the boundary of the split.

Assume without loss of generality that y^1 and y^2 are on L_1. Consider the family $\mathcal{T}$ of triangles whose edges are supported by L_2 and by two lines passing through y^1 and y^2 and whose interior contains the segment y^1y^2. See Figure 5.4. Note that all triangles in $\mathcal{T}$ are of Type 2. For $T \in \mathcal{T}$ we will denote by ψ_T the minimal function associated with T.

By assumption, $\sum_{i=1}^k \psi_S(r^i)\bar{s}_i < 1$. Let $\epsilon = 1 - \sum_{i=1}^k \psi_S(r^i)\bar{s}_i$.

We now make the following simple observation. Given a finite set X of points that lie in the interior of the split S, we can find a triangle $T \in \mathcal{T}$ as defined above, such that all points in X are in the interior of T. To see this, consider the convex hull $\mathcal{C}(X)$ of X. Since all points in X are in the interior of S, so is $\mathcal{C}(X)$. This implies that the tangent lines from y^1 and y^2 to $\mathcal{C}(X)$ are not parallel to L_1. Two of these four tangent lines along with L_2 of S form a triangle in $\mathcal{T}$ with X in its interior.

Let $s_{max} = \max\{\bar{s}_i \ : \ i = 1 \ldots, k\}$ and define $\delta = \frac{\epsilon}{2 \cdot k \cdot s_{max}} > 0$. For every ray r^i define $c(r^i) = \psi_S(r^i) + \delta$. Therefore, by definition $p^i = f + \frac{1}{c(r^i)} \cdot r^i$ is a point strictly in the interior of S. Using the observation stated above, there exists a triangle $T \in \mathcal{T}$ which contains all the points p^i. It follows that the coefficient $\psi_T(r^i)$ for any ray r^i is less than or equal to $c(r^i)$.

We claim that for this triangle T we have $\sum_{i=1}^k \psi_T(r^i)\bar{s}_i < 1$. Indeed,

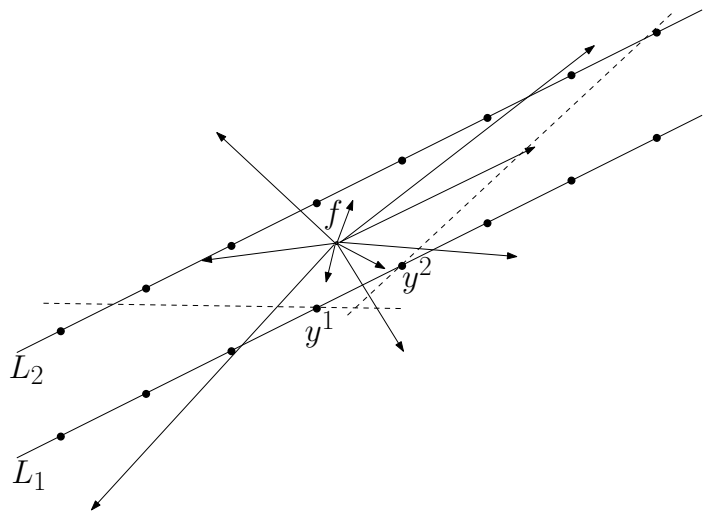


Figure 5.4: Approximating a split inequality with a triangle inequality. The triangle is formed by L_2 and the two dashed lines

$$\begin{aligned}
\sum_{i=1}^{k} \psi_T(r^i)\bar{s}_i &\leq \sum_{i=1}^{k} c(r^i)\bar{s}_i \\
&= \sum_{i=1}^{k} (\psi_S(r^i) + \delta)\bar{s}_i = \sum_{i=1}^{k} \psi_S(r^i)\bar{s}_i + \sum_{i=1}^{k} \frac{\epsilon}{2 \cdot k \cdot s_{max}}\bar{s}_i \\
&\leq \sum_{i=1}^{k} \psi_S(r^i)\bar{s}_i + \frac{\epsilon}{2} = 1 - \frac{\epsilon}{2} < 1
\end{aligned}$$

The first inequality follows from the definition of $c(r^i)$ and the last equality follows from the fact that $\sum_{i=1}^{k} \psi_S(r^i)\bar{s}_i = 1 - \epsilon$.

This shows that $T_f^k \subseteq S_f^k$. For the quadrilateral closure, we also use two lines passing through y^3 and y^4 on L_2 and argue similarly. □

Note that even though there can be a zero coefficient in a split inequality for some ray r, in the proof above we exhibit a sequence of triangle inequalities with arbitrarily small coefficients for ray r. Any point cut off by the split inequality is also cut off by a cut in the sequence.

5.2 Tools

5.2.1 Conditions for facets

Andersen, Louveaux, Weismantel and Wolsey [8] characterized the facets of R_f^k as arising from splits, triangles and quadrilaterals. Cornuéjols and Margot [34] gave a converse. We give this characterization in Theorem 70. Roughly speaking, for a maximal lattice-free triangle or quadrilateral to give rise to a facet, it has to have its corner points on half-lines $f + \lambda r^j$ for some $j = 1, \ldots, k$ and $\lambda > 0$; or to satisfy a certain technical condition called the ray condition. Although the ray condition is not central to the results in this Chapter (it is only used once in the proof of Theorem 82), we need to include it for technical completeness.

Let B_ψ be a maximal lattice-free split, triangle or quadrilateral with f in its interior. For any $j = 1, \ldots, k$ such that $\psi(r^j) > 0$, let p^j be the intersection of the half-line $f + \lambda r^j$, $\lambda \geq 0$, with the boundary of B_ψ. The point p^j is called the *boundary point* for r^j. Let P be a set of boundary points. We say that a point $p \in P$ is *active* if it can have a positive coefficient in a convex combination of points in P generating an integral point. Note that $p \in P$ is active if and only if p is integral or there exists $q \in P$ such that the segment pq contains an integral point in its interior. We say that an active point $p \in P$ is *uniquely active* if it has a positive coefficient in *exactly one* convex combination of points in P generating an integral point.

Apply the following *Reduction Algorithm*:

0.) Let $P = \{p^1, \ldots, p^k\}$.

1.) While there exists $p \in P$ such that p is active and p is a convex combination of other points in P, remove p from P. At the end of this step, P contains at most two active points on each edge of B_ψ and all points of P are distinct.

2.) While there exists a uniquely active $p \in P$, remove p from P.

3.) If P contains exactly two active points p and q (and possibly inactive points), remove both p and q from P.

The ray condition holds for a triangle or a quadrilateral if $P = \emptyset$ at termination of the Reduction Algorithm.

The ray condition holds for a split if, at termination of the Reduction Algorithm, either $P = \emptyset$, or $P = \{p_1, q_1, p_2, q_2\}$ with p_1, q_1 on one of the boundary lines and

p_2, q_2 on the other and both line segments p_1q_1 and p_2q_2 contain at least two integral points.

Theorem 70. (Cornuéjols and Margot [34]) *The facets of R_f^k are*

(i) split inequalities where the unbounded direction of B_ψ is r^j for some $j = 1, \ldots, k$ and the line $f + \lambda r^j$ contains no integral point; or where B_ψ satisfies the ray condition,

(ii) triangle inequalities where the triangle B_ψ has its corner points on three half-lines $f + \lambda r^j$ for some $j = 1, \ldots, k$ and $\lambda > 0$; or where the triangle B_ψ satisfies the ray condition,

(iii) quadrilateral inequalities where the corners of B_ψ are on four half-lines $f + \lambda r^j$ for some $j = 1, \ldots, k$ and $\lambda > 0$, and B_ψ satisfies a certain ratio condition (the ratio condition will not be needed in the results presented in this thesis; the interested reader is referred to [34] for details).

Note that the same facet may arise from different convex sets. For example quadrilaterals for which the ray condition holds define facets, but there is always also a triangle that defines the same facet, which is the reason why there is no mention of the ray condition in (iii) of Theorem 70.

5.2.2 Reducing the number of rays in the analysis

The following technical theorem will be used in the proofs of Theorems 67, 68 and 69, where we will be applying Theorem 64.

Theorem 71. *Let $B_1, \ldots, B_m$ be lattice-free convex sets with f in the interior of B_p, $p = 1, \ldots, m$. Let $R_c \subseteq \{1, \ldots, k\}$ be a subset of the ray indices such that for every ray r^j with $j \notin R_c$, r^j is the convex combination of some two rays in R_c. Define*

$$\begin{aligned} z_1 = \quad \min \quad & \sum_{i=1}^{k} s_i \\ & \sum_{i=1}^{k} \psi_{B_p}(r^i) s_i \quad \geq 1 \quad \textit{for} \;\; p = 1, \ldots, m \\ & s \geq 0 \end{aligned}$$

and

$$\begin{aligned} z_c = \quad \min \quad & \sum_{i \in R_c} s_i \\ & \sum_{i \in R_c} \psi_{B_p}(r^i) s_i \quad \geq 1 \quad \textit{for} \;\; p = 1, \ldots, m \\ & s \geq 0 \; . \end{aligned}$$

Then $z_1 = z_c$.

Proof. Assume there exists $j \notin R_c$ and r^1, r^2 are the rays in R_c such that $r^j = \lambda r^1 + (1-\lambda) r^2$ for some $0 < \lambda < 1$. Let $K = \{1, \ldots, k\} - j$ and define

$$\begin{aligned} z_2 = \quad \min \quad & \sum_{i \in K} s_i \\ & \sum_{i \in K} \psi_{B_p}(r^i) s_i \quad \geq 1 \quad \text{for} \;\; p = 1, \ldots, m \\ & s \geq 0 \; . \end{aligned}$$

We first show that $z_1 = z_2$. Applying the same reasoning repeatedly to all the indices not in R_c yields the proof of the theorem.

Any optimal solution for the LP defining z_2 yields a feasible solution for the one defining z_1 by setting $s_j = 0$, implying $z_1 \leq z_2$. It remains to show that $z_1 \geq z_2$.

Consider any point $\hat{s}$ satisfying $\sum_{i=1}^{k} \psi_{B_p}(r^i) \hat{s}_i \geq 1$ for every $p \in \{1, \ldots, m\}$. Consider the following values $\bar{s}$ for the variables corresponding to the indices $t \in K$.

$$\bar{s}_t = \begin{cases} \hat{s}_t & \text{if } \; t \notin \{1, 2, j\} \\ \hat{s}_1 + \lambda \hat{s}_j & \text{if } \; t = 1 \\ \hat{s}_2 + (1-\lambda) \hat{s}_j & \text{if } \; t = 2 \end{cases}$$

One can check that

$$\sum_{i \in K} \bar{s}_i = \hat{s}_j + \sum_{i \in K} \hat{s}_i \; .$$

From the results of Chapter 3, ψ_{B_p} is convex, thus $\psi_{B_p}(r^j) \leq \lambda \psi_{B_p}(r^1) + (1-\lambda) \psi_{B_p}(r^2)$ for $p = 1, \ldots, m$. It follows that $\sum_{i \in K} \psi_{B_p}(r^i) \bar{s}_i \geq \psi_{B_p}(r^j) \hat{s}_j + \sum_{i \in K} \psi_{B_p}(r^i) \hat{s}_i = \sum_{i=1}^{k} \psi_{B_p}(r^i) \hat{s}_i \geq 1$ for $p = 1, \ldots, m$. Hence $\bar{s}$ satisfies all the inequalities restricted to indices in K and has the same objective value as $\hat{s}$. It follows that $z_1 \geq z_2$. $\square$

5.3 Proof sketch for Theorems 67 and 68

In this section, we give a brief outline of the proofs of Theorems 67 and 68. A complete proof will be given in Sections 5.4 and 5.5 respectively.

In Theorem 67, we need to analyze the optimization problem

$$\min \left\{ \sum_{i=1}^{k} \psi(r^i) s_i \ : \ s \in S_f^k \right\} , \tag{5.2}$$

where ψ is the minimal function derived from the Type 1 triangle.

For Theorem 68, recall that all nontrivial facet defining inequalities for R_f^k are of the form $a^i s \geq 1$ with $a^i \geq 0$. Therefore, Theorem 64 shows that to prove Theorem 68, we need to consider all nontrivial facet defining inequalities and optimize in the direction a^i over the triangle closure T_f^k and the quadrilateral closure Q_f^k. This task is made easier since all the nontrivial facets of R_f^k are characterized in Theorem 70. Moreover, Theorem 65 shows that every split inequality $\sum_{i=1}^{k} \psi(r^i) s_i \geq 1$ is valid for T_f^k. Therefore, if we optimize over T_f^k in the direction $\sum_{i=1}^{k} \psi(r^i) s_i$, we get a value of at least one. Theorem 65 shows that this also holds for Q_f^k. Thus, we can ignore the facets defined by split inequalities.

Formally, consider a maximal lattice-free triangle or quadrilateral B with associated minimal function ψ that gives rise to a facet $\sum_{j=1}^{k} \psi(r^j) s_j \geq 1$ of R_f^k. We want to investigate the following optimization problems:

$$\inf \left\{ \sum_{j=1}^{k} \psi(r^j) s_j \ : \ s \in T_f^k \right\} \tag{5.3}$$

and

$$\inf \left\{ \sum_{j=1}^{k} \psi(r^j) s_j \ : \ s \in Q_f^k \right\} . \tag{5.4}$$

We first observe that, without loss of generality, we can make the following simplifying assumptions for problems (5.2), (5.3) and (5.4).

Assumption 72. *Consider the objective function ψ in problems (5.2), (5.3) and (5.4). For every j such that $\psi(r^j) > 0$, the ray r^j is such that the point $f + r^j$ is on the*

boundary of the lattice-free set B generating ψ.

Indeed, this amounts to scaling the coefficient for the ray r^j by a constant factor in every inequality derived from all maximal lattice-free sets, including B. Therefore, this corresponds to a simultaneous scaling of variable s_j and corresponding coefficients in problems (5.2), (5.3) and (5.4). This does not change the optimal values of these problems. Moreover, Cornuéjols and Margot [34] show that the equations of all edges of triangles of Type 1, 2, or 3, of quadrilaterals and the direction of all splits generating facets of R_f^k are rational. This implies that the scaling factor for ray r^j is a rational number and that the scaled ray is rational too.

As a consequence, we can assume that the objective function of problems (5.2), (5.3) or (5.4) is $\sum_{j=1}^{k} s_j$.

When B_ψ is a triangle or quadrilateral and f is in its interior, define a *corner ray* to be a ray r such that $f + \lambda r$ is a corner of B_ψ for some $\lambda > 0$.

Remark 73. *If $\{r^1, \ldots, r^k\}$ contains the corner rays of the convex set defining the objective functions of (5.2), (5.3) or (5.4), then Assumption 72 implies that the hypotheses of Theorem 71 are satisfied. Therefore, when analyzing (5.2), (5.3) or (5.4), we can assume that $\{r^1, \ldots, r^k\}$ is exactly the set of corner rays.*

5.4 Type 1 triangle and the split closure

In this section, we present the proof of Theorem 67.

Consider any Type 1 triangle T with integral vertices x^j, for $j = 1, 2, 3$, and one integral point y^j for $j = 1, 2, 3$ in the interior of each edge. We want to study the optimization problem (5.2). Recall that Remark 73 says that we only need to consider the case with three corner rays r^1, r^2 and r^3.

We compute the exact value of

$$\begin{aligned} z_{SPLIT} = \quad \min \quad & \sum_{j=1}^{3} s_j \\ & \sum_{j=1}^{3} \psi(r^j) s_j \quad \geq 1 \quad \text{for all splits } B_\psi \\ & s \in \mathbb{R}^3_+. \end{aligned} \tag{5.5}$$

Observe that, using a unimodular transformation, T can be made to have one horizontal edge x^1x^2 and one vertical edge x^1x^3, as shown in Figure 5.3. Without loss of generality, we place the origin at point x^1.

We distinguish two cases depending on the position of f in the interior of triangle T: f is in the inner triangle T_I with vertices $y^1 = (1,1)$, $y^2 = (0,1)$ and $y^3 = (1,0)$; and $f \in int(T) \setminus T_I$. We show that $z_{SPLIT} = \frac{1}{2}$ when f is in the inner triangle T_I and that z_{SPLIT} increases linearly from $\frac{1}{2}$ when f is at the boundary of T_I to $\frac{2}{3}$ at the corners of the triangle T when $f \in int(T) \setminus T_I$. See the right part of Figure 5.3 for some level curves of z_{SPLIT} as a function of the position of f in T. By a symmetry argument, it is sufficient to consider the inner triangle T_I and the corner triangle T_C defined by $f_1 + f_2 \leq 1$, $f_1, f_2 \geq 0$.

Theorem 74. *Let T be a Type 1 triangle with integral vertices, say $(0,0)$, $(0,2)$ and $(2,0)$. Then*

(i) $z_{SPLIT} = \frac{1}{2}$ *when f is interior to the triangle with vertices $(1,0)$, $(0,1)$ and $(1,1)$.*

(ii) $z_{SPLIT} = 1 - \frac{1}{3-f_1-f_2}$ *when $f = (f_1, f_2)$ is interior to the corner triangle $f_1 + f_2 \leq 1$, $f_1, f_2 \geq 0$. The value of z_{SPLIT} when f is in the other corner triangles follows by symmetry.*

To prove this theorem, we show that the split closure is completely defined by only three split inequalities. In other words, all other split inequalities are dominated by these three split inequalities.

Define S_1 as the convex set $1 \leq x_1 + x_2 \leq 2$, S_2 as the convex set $0 \leq x_1 \leq 1$ and S_3 as the convex set $0 \leq x_2 \leq 1$. Define *Split 1* (resp. *Split 2*, *Split3*) to be the inequality obtained from S_1 (resp. S_2, S_3).

Let S be a split $c \leq ax_1 + ba_2 \leq c + 1$ with f in the interior of S. The *shores* of S are the two half-planes $ax_1 + ba_2 \leq c$ and $ax_1 + ba_2 \geq c + 1$.

Remark 75. *Let A, B, and C be three points on a line, with B between A and C and let S be a split. If A and C are not in the interior of S but B is, then A and C are on opposite shores of S.*

Lemma 76. *If f is in the interior of the triangle T_I with vertices $(0,1)$, $(1,0)$ and $(1,1)$, then the split closure is defined by Split 1, Split 2 and Split 3.*

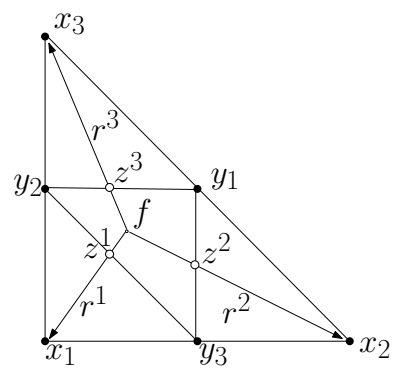


Figure 5.5: Illustration for the proof of Lemma 76

Proof. Let S be a split defining an inequality that is not dominated by either of Split 1, Split 2, or Split 3. For $i = 1, 2, 3$, let z^i (resp. w^i) be the boundary point for r^i on the boundary of T_I (resp. S) (see Figure 5.5). (Note that since $x^i = f + r^i$ is integer, these points exist.) Observe that if z^1 is not in the interior of S, then the inequality obtained from S is dominated by Split 1, since the three boundary points w_1, w_2, w_3 are closer to f than the corresponding three boundary points z^1, x^2, x^3 for the three rays on the boundary of S_1. A similar observation holds for z^2 and Split 2 and for z^3 and Split 3, yielding that z^2 and z^3 are also in the interior of S.

Since the points y^1, y^2 and y^3 are integer, they are not in the interior of S. Applying Remark 75 to y^1, z^3, y^2, we must have that y^1 and y^2 are on opposite shores of S. Now, y^3 is in one of the two shores of S. Assume without loss of generality that it is on the same shore as y^1. Applying Remark 75 to y^1, z^2, y^3, we have that y^1 and y^3 are on opposite shores of S, a contradiction. □

Lemma 77. *If f is in the interior of the triangle T_I with vertices $(0, 1)$, $(1, 0)$ and $(1, 1)$, then $z_{SPLIT} = \frac{1}{2}$.*

Proof. By Lemma 76, z_{SPLIT} is given by

$$\begin{aligned} z_{SPLIT} = \quad \min \quad & \sum_{j=1}^{3} s_j \\ & \sum_{j=1}^{3} \psi_i(r^j) s_j \quad \geq 1 \quad \text{for } i = 1, 2, 3 \\ & s \in \mathbb{R}^3_+ \end{aligned} \tag{5.6}$$

where $\psi_i(r^j)$ is the coefficient of s_j in Split i, for $i = 1, 2, 3$. Let $f = (f_1, f_2)$. The coefficient of s^j in the split inequality can be computed from the boundary point for r^j with the corresponding split. For example, the boundary points for r^2 and r^3 with S_1 are the integer points x^2 and x^3. This implies that $\psi_1(r^2) = \psi_1(r^3) = 1$. On the other hand, the boundary point for r^1 is the point $t = \left(\frac{f_1}{f_1+f_2}, \frac{f_2}{f_1+f_2}\right)$. The length of r^1 divided by the length of the segment ft determines the coefficient $\psi_1(r^1)$ of s_1 (This follows from the homogeneity of ψ_1 and the fact that $\psi_1(t - f) = 1$ since t is on the boundary of S_1). We get $\psi_1(r^1) = \frac{f_1+f_2}{f_1+f_2-1}$. Repeating this for S_2 and S_3, we get that z_{SPLIT} is the optimal value of the following linear program.

$$
\begin{array}{rlrrrl}
z_{SPLIT} = & \min & s_1 & +s_2 & +s_3 & \\
 & & \frac{f_1+f_2}{f_1+f_2-1}s_1 & +s_2 & +s_3 & \geq 1 \\
 & & s_1 & +\frac{2-f_1}{1-f_1}s_2 & +s_3 & \geq 1 \\
 & & s_1 & +s_2 & +\frac{2-f_2}{1-f_2}s_3 & \geq 1 \\
 & & & s \geq 0. & &
\end{array}
\tag{5.7}
$$

Its optimal solution s^* is $s_1^* = \frac{f_1+f_2-1}{2}$, $s_2^* = \frac{1-f_1}{2}$, $s_3^* = \frac{1-f_2}{2}$ with value $z_{SPLIT} = s_1^* + s_2^* + s_3^* = \frac{1}{2}$. Indeed, note that all three inequalities in (5.7) are satisfied at equality and that the dual of (5.7) is feasible (for example, $(0, 0, 0)$ is a solution). Therefore the complementary slackness conditions hold for s^* with any feasible solution of the dual.

□

Now we prove the second part of the theorem, when f is interior to the corner triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ or an inner point on the segment y^2y^3.

Lemma 78. *If f is in the interior of the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$, or an inner point on the segment joining $(0, 1)$ to $(1, 0)$, then the split closure is defined by Split 2 and Split 3.*

Proof. Let S be a split defining a split inequality that is not dominated by either of Split 2, or Split 3. Let z^2 be the intersection point of r^2 with y^1y^3 and let z^3 be the intersection point of r^3 with y^1y^2. For $i = 1, 2, 3$, let w^i be the intersection point of r^i with either L_1 or L_2. (Note that since x^i is integer, r^i has to intersect one of the two lines.) Observe that if z^2 is not in the interior of S, then the inequality obtained from S is dominated by Split 2, since the three intersection points w_1, w_2, w_3 are closer to f than the corresponding three intersection points x^1, z^2, x^3 for S_2. A similar observation holds for z^3 and S_3, yielding that z^3 is also in the interior of S.

Since the points y^1, y^2 and y^3 are integer, they are not in the interior of S. Applying Remark 75 to y^1, z^3, y^2, we have that y^1 and y^2 are on opposite shores of S. Applying Remark 75 to y^1, z^2, y^3, we have that y^1 and y^3 are on opposite shores of S. It follows that y^2 and y^3 are on the same shore W of S and thus the whole segment y^2y^3 is in W. This is a contradiction with the fact that both f and z^3 are in the interior of S, as the two segments y^2y^3 and fz^3 intersect. □

Lemma 79. *If $f = (f_1, f_2)$ is in the interior of the triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$, or an inner point on the segment joining $(0,1)$ to $(1,0)$, then $z_{SPLIT} = 1 - \frac{1}{3-f_1-f_2}$.*

Proof. The optimal solution of the LP

$$\begin{array}{rrrrl} z_{SPLIT} = & \min s_1 & +s_2 & +s_3 & \\ & s_1 & +\frac{2-f_1}{1-f_1}s_2 & +s_3 & \geq 1 \\ & s_1 & +s_2 & +\frac{2-f_2}{1-f_2}s_3 & \geq 1 \\ & & s \geq 0. & & \end{array} \tag{5.8}$$

is $s_1 = 0$, $s_2 = \frac{1-f_1}{3-f_1-f_2}$, $s_3 = \frac{1-f_2}{3-f_1-f_2}$. □

This completes the proof of Theorem 74. This theorem in conjunction with Theorem 64 implies that including all Type 1 triangle facets can improve upon the split closure only by a factor of 2.

Corollary 80. *Let $\mathcal{F}$ be the family of all facet defining inequalities arising from Type 1 triangles. Define*

$$\bar{S}_f = S_f^k \cap \left\{ \sum_{i=1}^{k} \psi(r^i) s_i \geq 1 \ : \ \psi \in \mathcal{F} \right\} .$$

Then $\bar{S}_f \subseteq S_f^k \subseteq 2\bar{S}_f$.

5.5 Integer hull vs. triangle and quadrilateral closures

In this section we present the proof of Theorem 68. We show that the triangle closure T_f^k and the quadrilateral closure Q_f^k both approximate the integer hull R_f^k

to within a factor of two. As outlined in Section 5.3, we can show this by taking a facet of R_f^k, and optimizing in that direction over T_f^k or Q_f^k. As noted in that section, we need to analyze the optimization problems (5.3) and (5.4), where the objective function comes from a maximal lattice-free triangle or quadrilateral.

5.5.1 Approximating the integer hull by the triangle closure

We only need to consider facets of R_f^k derived from quadrilaterals to obtain the objective function of problem (5.3). We prove the following result.

Theorem 81. *Let Q be a maximal lattice-free quadrilateral with corresponding minimal function ψ and generating a facet $\sum_{i=1}^{k} \psi(r^i)s_i \geq 1$ of R_f^k. Then*

$$\inf \left\{ \sum_{i=1}^{k} \psi(r^i)s_i \ : \ s \in T_f^k \right\} \geq \frac{1}{2} \ .$$

Proof. The theorem holds if the facet defining inequality can also be obtained as a triangle inequality. Therefore by Theorem 70, we may assume that rays $r^1, \ldots r^4$ are the corners rays of Q (see Figure 5.6). We remind the reader of Remark 73, showing that we can assume that $k = 4$ and that the four rays are exactly the corner rays of Q.

By a unimodular transformation, we may further assume that the four integral points on the boundary of Q are $(0,0), (1,0), (1,1), (0,1)$. Moreover, by symmetry, we may assume that the fractional point f satisfies $f_1 \leq \frac{1}{2}$ and $f_2 \leq \frac{1}{2}$ as rotating this region about $(\frac{1}{2}, \frac{1}{2})$ by multiples of $\frac{\pi}{2}$ covers the entire quadrilateral. Note that $f_1 < 0$ and $f_2 < 0$ are possible.

We relax Problem (5.3) by keeping only two of the triangle inequalities, defined by triangles T_1 and T_2. T_1 has corner $f + r^4$ and edges supported by the two edges of Q incident with that corner and by the line $x = 1$. T_2 has corner $f + r^1$ and edges supported by the two edges of Q incident with that corner and by the line $y = 1$. The two triangles are depicted in dashed lines in Figure 5.6.

Thus, Problem (5.3) can be relaxed to the LP

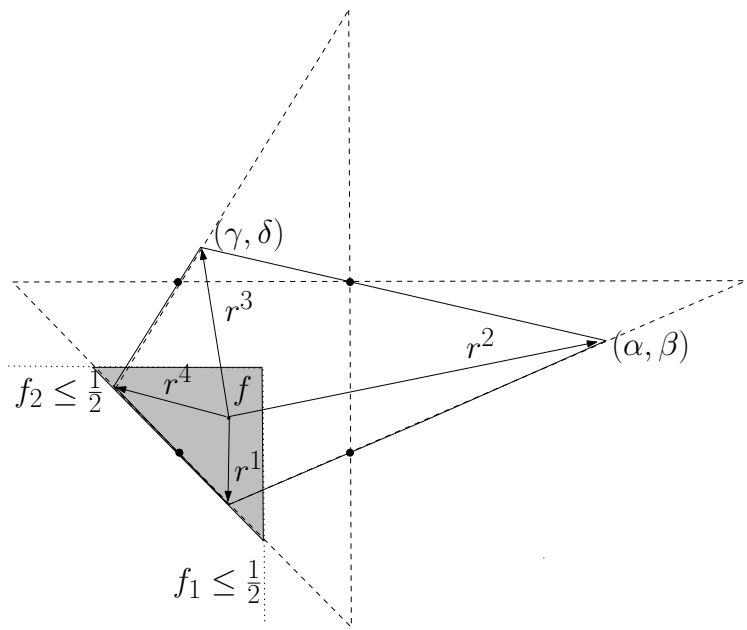


Figure 5.6: Approximating a quadrilateral inequality with triangle inequalities

$$
\begin{array}{lll}
\min & s_1 + s_2 + s_3 + s_4 & \\
& \sum_{i=1}^{4} \psi_{T_1}(r^i) s_i & \geq 1 \quad \text{(Triangle } T_1\text{)} \\
& \sum_{i=1}^{4} \psi_{T_2}(r^i) s_i & \geq 1 \quad \text{(Triangle } T_2\text{)} \\
& s \in \mathbb{R}^4_+. &
\end{array}
\tag{5.9}
$$

Let $(\alpha, \beta) = f + r^2$ and $(\gamma, \delta) = f + r^3$. Computing the coefficients $\psi_{T_1}(r^2)$ and $\psi_{T_2}(r^3)$, LP (5.9) becomes

$$
\begin{array}{lll}
\min & s_1 + s_2 + s_3 + s_4 & \\
& s_1 + \dfrac{\alpha - f_1}{1 - f_1} s_2 + s_3 + s_4 & \geq 1 \quad (T_1) \\
& s_1 + s_2 + \dfrac{\delta - f_2}{1 - f_2} s_3 + s_4 & \geq 1 \quad (T_2) \\
& s \in \mathbb{R}^4_+. &
\end{array}
\tag{5.10}
$$

Using the equation of the edge of Q connecting $f + r^2$ and $f + r^3$, we can find bounds on $\psi_{T_1}(r^2)$ and $\psi_{T_2}(r^3)$. The edge has equation $x_1\frac{1}{t} + \frac{t-1}{t}x_2 = 1$ for some $1 < t < \infty$. Therefore $\alpha \leq t$ and $\delta \leq \frac{t}{t-1}$. Using these two inequalities together with $f_1 \leq \frac{1}{2}$ and $f_2 \leq \frac{1}{2}$ we get

$$\frac{\alpha - f_1}{1 - f_1} = \frac{\alpha - 1}{1 - f_1} + 1 \leq 2(t-1) + 1 = 2t - 1 \qquad \text{and} \qquad \frac{\delta - f_2}{1 - f_2} \leq 2\frac{t}{t-1} - 1\ .$$

Using these bounds, we obtain the relaxation of LP (5.10)

$$\begin{array}{lll} \min & s_1 + s_2 + s_3 + s_4 & \\ & s_1 + (2t-1)s_2 + s_3 + s_4 & \geq 1 \quad (T_1) \\ & s_1 + s_2 + (2\dfrac{t}{t-1} - 1)s_3 + s_4 & \geq 1 \quad (T_2) \\ & s \in \mathbb{R}^4_+. & \end{array} \tag{5.11}$$

Set $\lambda = 2t - 1$ and $\mu = 2\frac{t}{t-1} - 1$. Then $t > 1$ implies $\lambda > 1$ and $\mu > 1$. The optimal solution of the above LP is $s_1 = s_4 = 0$, $s_2 = \frac{\mu-1}{\lambda\mu-1}$ and $s_3 = \frac{\lambda-1}{\lambda\mu-1}$ with value

$$s_1 + s_2 + s_3 + s_4 = \frac{\lambda + \mu - 2}{\lambda\mu - 1} = \frac{t^2 - 2t + 2}{t^2}\ .$$

To find the minimum of this expression for $t > 1$, we set its derivative to 0, and get the solution $t = 2$. Thus the minimum value of $s_1 + s_2 + s_3 + s_4$ is equal to $\frac{1}{2}$.

□

5.5.2 Approximating the integer hull by the quadrilateral closure

In this section, we study Problem (5.4). We can approximate the facets derived from Type 1 and Type 2 triangles using quadrilaterals in a similar manner as the splits were approximated by triangles and quadrilaterals. See Figure 5.7. We again define the set X of points which lie strictly inside the Type 1 or Type 2 triangle, similar to the proof of Theorem 65. Then we can find quadrilaterals as shown in Figure 5.7 that contain the set X. The proof goes through in exactly the same manner.

However triangles of Type 3 pose a problem. They cannot be approximated to any desired precision by a sequence of quadrilaterals.

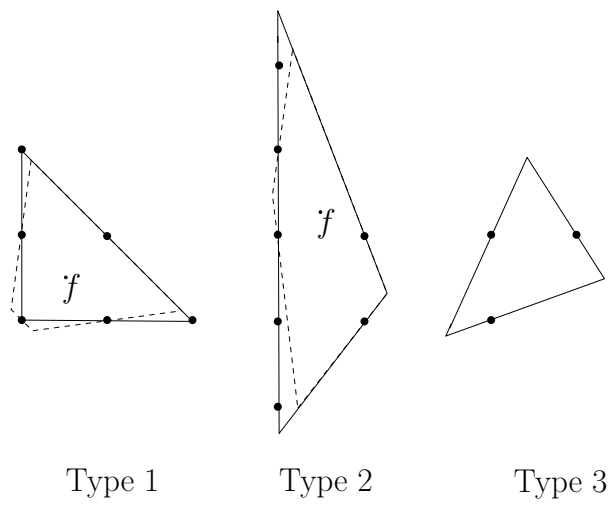


Figure 5.7: Approximating a triangle inequality with a quadrilateral inequality

In this section, we work under Assumption 72. By Theorem 70, a Type 3 triangle T defines a facet if and only if either the ray condition holds, or all three corner rays are present. First we consider the case where the ray condition holds.

Theorem 82. *Let T be a triangle of Type 3 with corresponding minimal function ψ_T defining a facet of R^k_f. If the ray condition holds for T then Problem (5.4) has optimal value 1.*

Proof. We first prove that if the ray condition holds the points $p^j = f + r^j$ are integral points on the boundary of T, for $j = 1, \ldots, k$.

For $i = 1, 2, 3$, let P_i be the set of points left at the end of Step i of the Reduction Algorithm given in Section 5.2. Each $p^j \in P_1$ with p^j integral is uniquely active and is removed during Step 2 of the Reduction Algorithm. Hence, all points in P_2 are non-integral. Observe that Step 3 can only remove boundary points p and q when the segment pq contains at least two integral points in its relative interior. Therefore this step does not remove anything in a Type 3 triangle and $P_3 = P_2$.

Since the ray condition holds, we have $P_3 = P_2 = \emptyset$ and P_1 contains only integral points. But then $P_1 = P$, showing that all boundary points at the beginning of the Reduction Algorithm are integral.

It is then straightforward to construct a maximal lattice-free quadrilateral Q with p^j, $j = 1, \ldots, k$ on its boundary, and containing f in its interior. It follows that the value of Problem (5.4) is equal to 1. □

We now consider the case where T is a Type 3 triangle with the three corner rays present. In this case, we can approximate the facet obtained from T to within a factor of two by using inequalities derived from triangles of Type 2. Define another relaxation $\bar{T}_f^k$ as the convex set defined by the intersection of the inequalities derived only from Type 1 and Type 2 triangles. By definition, $T_f^k \subseteq \bar{T}_f^k$. From the discussion at the beginning of this section, we also know $Q_f^k \subseteq \bar{T}_f^k$. Hence (5.4) can be relaxed to

$$\inf \left\{ \sum_{i=1}^{k} \psi(r^i) s_i : \ s \in \bar{T}_f^k \right\} . \tag{5.12}$$

Theorem 83. *Let T be a triangle of Type 3 with corresponding minimal function ψ and generating a facet $\sum_{i=1}^{k} \psi(r^i) s_i \geq 1$ of R_f^k. Then,*

$$\inf \left\{ \sum_{i=1}^{k} \psi(r^i) s_i \ : \ s \in \bar{T}_f^k \right\} \geq \frac{1}{2} .$$

This theorem implies directly the following corollary.

Corollary 84. $Q_f^k \subseteq 2R_f^k$.

Proof of Theorem 83. We first make an affine transformation to simplify computations. Let y^1, y^2, y^3 be the three lattice points on the sides of T. We choose the transformation such that the two following properties are satisfied.

(i) The standard lattice is mapped to the lattice generated by the vectors $v^1 = (1, 0)$ and $v^2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, i.e. all points of the form $z_1 v^1 + z_2 v^2$, where z_1, z_2 are integers.

(ii) y^1, y^2, y^3 are respectively mapped to $(0, 0), (1, 0), (0, 1)$ in the above lattice.

We use the basis v^1, v^2 for $\mathbb{R}^2$ for all calculations and equations in the remainder of the proof. See Figure 5.8.

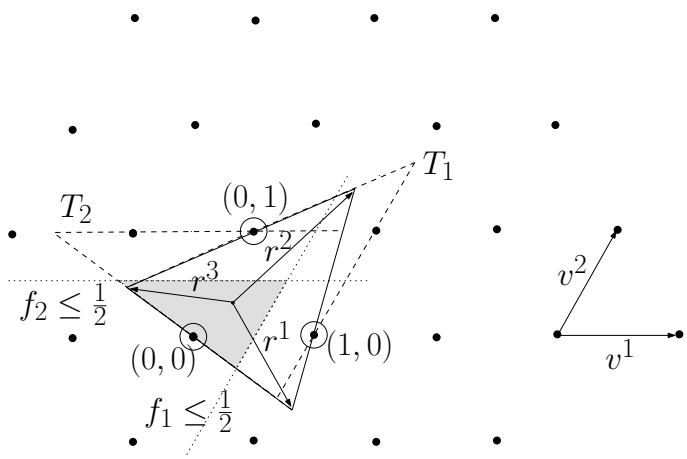


Figure 5.8: Approximating a Type 3 triangle inequality with Type 2 triangle inequalities. The Type 3 triangle is in solid lines. The basis vectors are $v^1 = (1,0)$ and $v^2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$

With this transformation, we can get a simple characterization for the Type 3 triangles. See Figure 5.8 for an example. We make the following claim about the relative orientations of the three sides of the Type 3 triangle.

Lemma 85. *Let $\mathcal{F}$ be the family of triangles formed by three lines given by:*

$$\begin{aligned}
&\textit{Line } 1: \quad -\frac{x_1}{t_1} + x_2 = 1 \quad \textit{with} \;\; 0 < t_1 < \infty \;; \\
&\textit{Line } 2: \quad t_2\, x_1 + x_2 = 0 \quad \textit{with} \;\; 0 < t_2 < 1 \;; \\
&\textit{Line } 3: \quad x_1 + \frac{x_2}{t_3} = 1 \quad \textit{with} \;\; 1 < t_3 < \infty \;.
\end{aligned}$$

Any Type 3 triangle is either a triangle from $\mathcal{F}$ or a reflection of a triangle from $\mathcal{F}$ about the line $x_1 = x_2$.

Proof. Take any Type 3 triangle T. Consider the edge passing through $(1,0)$. Since it cannot go through the interior of the equilateral triangle $(0,0),(0,1),(1,0)$, there are only two choices for its orientation : a) It can go through the segment $(0,1),(1,1)$, or b) It can go through the segment $(0,0),(1,-1)$.

In the first case its equation is that of Line 3. This now forces the edge of T passing through $(0,0)$ to have the equation for Line 2. This is because the only

other possibility for this edge would be for the line to pass through the segment $(-1,1),(0,1)$. But then the lattice point $(1,-1)$ is included in the interior of the triangle. Similarly, the third edge must now have Line 1's equation, because $(-1,1)$ needs to be excluded from the interior.

Case b) can be mapped to Case a) by a reflection about the line $x_1 = x_2$. □

Remark 86. *We can choose any values for t_1, t_2, t_3 independently in the prescribed ranges, and we get a lattice free triangle. This observation shows that the family $\mathcal{F}$ defined above is exactly the family of all Type 3 triangles modulo an affine transformation.*

We now show how to bound Problem (5.12) and hence prove Theorem 69.

Consider any Type 3 triangle T. It is sufficient to consider the case where the lines supporting the edges of T have equations as stated in the Lemma 85.

We consider two cases for the position of the fractional point $f = (f_1, f_2)$.

(i) $f_1 \leq \frac{1}{2}$, $f_2 \leq \frac{1}{2}$;

(ii) $f_1 \leq 0$, $f_1 + f_2 \leq \frac{1}{2}$.

The union of the two regions described above when rotated by $2\pi/3$ and $4\pi/3$ about the point $(\frac{1}{3}, \frac{1}{3})$ (the centroid of the triangle formed by (0,0),(0,1) and (1,0)), cover all of $\mathbb{R}^2$. Hence they cover all of T and by rotational symmetry, investigating these two cases is enough.

For the first case, we relax Problem (5.12) by using only two inequalities from $\bar{T}_f^k$. These are derived from Type 2 triangles T_1 and T_2 (see Figure 5.8), which are defined as follows. T_1 has Line 1 and Line 2 supporting two of its edges and $x_1 = 1$ supporting the third one (with more than one integral point). T_2 has Line 2 and Line 3 supporting two of its edges and $x_2 = 1$ supporting the third one (with more than one integral point). Let ψ_{T_1} and ψ_{T_2} be the corresponding minimal functions derived from T_1 and T_2.

The following LP is a relaxation of Problem (5.12).

$$\begin{array}{lll} \min & \sum_{i=1}^{k} s_i & \\ & \sum_{i=1}^{k} \psi_{T_1}(r^i) s_i & \geq 1 \quad \text{(Triangle } T_1\text{)} \\ & \sum_{i=1}^{k} \psi_{T_2}(r^i) s_i & \geq 1 \quad \text{(Triangle } T_2\text{)} \\ & s \in \mathbb{R}^k_+. & \end{array} \tag{5.13}$$

Theorem 71 and Remark 73 imply that LP (5.13) is equivalent to

$$\begin{array}{lll} \min & s_1 + s_2 + s_3 & \\ & \psi_{T_1}(r^1)s_1 + \psi_{T_1}(r^2)s_2 + \psi_{T_1}(r^3)s_3 & \geq 1 \quad \text{(Triangle } T_1\text{)} \\ & \psi_{T_2}(r^1)s_1 + \psi_{T_2}(r^2)s_2 + \psi_{T_2}(r^3)s_3 & \geq 1 \quad \text{(Triangle } T_2\text{)} \\ & s \in \mathbb{R}^3_+. & \end{array} \tag{5.14}$$

where r^1, r^2 and r^3 are the three corner rays (see Figure 5.8).

We now show that this LP has an optimal value of at least $\frac{1}{2}$.

Note that $\psi_{T_1}(r^2) = \psi_{T_1}(r^3) = 1$. $\psi_{T_1}(r^1)$ needs to be computed. First we compute r^1 and r^2 in terms of t_1, t_2, t_3, f_1 and f_2.

The intersection of Line 2 and Line 3 is given by

$$\left(\frac{t_3}{t_3 - t_2}, \frac{-t_2 t_3}{t_3 - t_2}\right) \qquad \text{and thus} \qquad r^1 = \left(\frac{t_3}{t_3 - t_2}, \frac{-t_2 t_3}{t_3 - t_2}\right) - (f_1, f_2)\ .$$

As $\psi_{T_1}(r^1) = \frac{1}{\gamma}$ where γ is such that $(f_1, f_2) + \gamma r^1$ lies on the line $x_1 = 1$, we get

$$\psi_{T_1}(r^1) = \frac{\frac{t_3}{t_3 - t_2} - f_1}{1 - f_1}\ .$$

Similarly, we only need $\psi_{T_2}(r^2)$ as $\psi_{T_2}(r^1) = \psi_{T_2}(r^3) = 1$. The intersection of Line 1 and Line 3 is

$$\left(\frac{t_1(t_3 - 1)}{1 + t_1 t_3}, \frac{t_3(t_1 + 1)}{1 + t_1 t_3}\right) \qquad \text{and thus} \qquad r^2 = \left(\frac{t_1(t_3 - 1)}{1 + t_1 t_3}, \frac{t_3(t_1 + 1)}{1 + t_1 t_3}\right) - (f_1, f_2)\ .$$

Computing the coefficient like before, we get

$$\psi_{T_2}(r^2) = \frac{\frac{t_3(t_1+1)}{1+t_1t_3} - f_2}{1-f_2} \ .$$

Hence LP (5.14) becomes

$$\begin{array}{lll} \min & s_1 + s_2 + s_3 & \\ & \dfrac{\frac{t_3}{t_3-t_2} - f_1}{1-f_1} s_1 + s_2 + s_3 & \geq 1 \quad \text{(Triangle } T_1\text{)} \\ & s_1 + \dfrac{\frac{t_3(t_1+1)}{1+t_1t_3} - f_2}{1-f_2} s_2 + s_3 & \geq 1 \quad \text{(Triangle } T_2\text{)} \\ & s \in \mathbb{R}^3_+. & \end{array} \tag{5.15}$$

As

$$\frac{\frac{t_3}{t_3-t_2} - f_1}{1-f_1} = \frac{\frac{t_3}{t_3-t_2} - 1}{1-f_1} + 1 \ ,$$

the assumptions $f_1 \leq \frac{1}{2}$ and $t_2 < 1$ yield

$$\frac{\frac{t_3}{t_3-t_2} - 1}{1-f_1} + 1 \leq 2\frac{t_3}{t_3-1} - 1 \ .$$

Similarly,

$$\frac{\frac{t_3(t_1+1)}{1+t_1t_3} - f_2}{1-f_2} = \frac{\frac{t_3(t_1+1)}{1+t_1t_3} - 1}{1-f_2} + 1$$

and the assumption $f_2 \leq \frac{1}{2}$ gives

$$\frac{\frac{t_3(t_1+1)}{1+t_1t_3} - 1}{1-f_2} + 1 \leq 2\frac{t_3(t_1+1)}{1+t_1t_3} - 1 = 2\frac{t_3-1}{1+t_1t_3} + 1 \ .$$

Under the assumption $t_1 > 0$, we obtain $2\frac{t_3-1}{1+t_1t_3} + 1 \leq 2t_3 - 1$.

To get a lower bound on (5.15), we thus can relax its constraints to

$$\begin{array}{lll} \min & s_1 + s_2 + s_3 & \\ & (2\dfrac{t_3}{t_3-1} - 1)s_1 + s_2 + s_3 & \geq 1 \quad \text{(Triangle } T_1\text{)} \\ & s_1 + (2t_3 - 1)s_2 + s_3 & \geq 1 \quad \text{(Triangle } T_2\text{)} \\ & s \in \mathbb{R}^3_+. & \end{array} \tag{5.16}$$

Set $\lambda = 2t_3 - 1$ and $\mu = 2\frac{t_3}{t_3-1} - 1$. Then $t_3 > 1$ implies $\lambda > 1$ and $\mu > 1$. The optimal solution of LP (5.16) is $s_1 = \frac{\lambda-1}{\lambda\mu-1}$, $s_2 = \frac{\mu-1}{\lambda\mu-1}$ and $s_3 = 0$ with value

$$s_1 + s_2 + s_3 = \frac{\lambda + \mu - 2}{\lambda\mu - 1} = \frac{t_3^2 - 2t_3 + 2}{t_3^2} .$$

To find the minimum over all $t_3 > 1$, we set the derivative to 0, which gives the solution $t_3 = 2$. Thus the minimum value of $s_1 + s_2 + s_3$ is $\frac{1}{2}$.

Next we consider $f_1 \leq 0$ and $f_1 + f_2 \leq \frac{1}{2}$, the shaded region in Figure 5.9. We relax Problem (5.12) using only two inequalities. We take T_1 as before, but T_2 is the triangle formed by the following three lines : Line 2 from Lemma 85, line parallel to Line 1 from Lemma 85 but passing through $(-1, 1)$ and the line passing through $(1, 0), (0, 1)$.

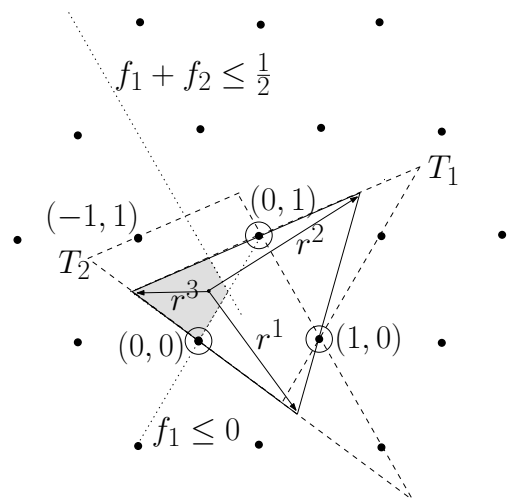


Figure 5.9: Approximating a Type 3 triangle inequality with Type 2 triangles inequalities - Case 2

As in the previous case, we formulate the relaxation as an LP with constraints corresponding to T_1 and T_2. The only difference from LP (5.14) is the coefficient $\psi_{T_2}(r^2)$. This time the point $(f_1, f_2) + \gamma r^2$ lies on the line $x_1 + x_2 = 1$ (recall that $\psi_{T_2}(r^2) = \frac{1}{\gamma}$). This gives us

$$\psi_{T_2}(r^2) = \frac{\frac{2t_1t_3+t_3-t_1}{1+t_1t_3} - f_1 - f_2}{1 - f_1 - f_2} .$$

We then have the following LP.

$$\begin{array}{lll} \min & s_1+s_2+s_3 & \\ & \dfrac{\frac{t_3}{t_3-t_2}-f_1}{1-f_1}s_1+s_2+s_3 & \geq 1 \quad \text{(Triangle } T_1\text{)} \\ & s_1+\dfrac{\frac{2t_1t_3+t_3-t_1}{1+t_1t_3}-f_1-f_2}{1-f_1-f_2}s_2+s_3 & \geq 1 \quad \text{(Triangle } T_2\text{)} \\ & s\in\mathbb{R}^3_+. & \end{array} \tag{5.17}$$

We simplify the coefficients as earlier :

$$\frac{\frac{t_3}{t_3-t_2}-f_1}{1-f_1}=1+\frac{\frac{t_3}{t_3-t_2}-1}{1-f_1} \qquad \text{and} \qquad \frac{\frac{2t_1t_3+t_3-t_1}{1+t_1t_3}-f_1-f_2}{1-f_1-f_2}=1+\frac{\frac{2t_1t_3+t_3-t_1}{1+t_1t_3}-1}{1-f_1-f_2}\ .$$

Using the assumptions $f_1\leq 0, f_1+f_2\leq\frac{1}{2}$, we get that

$$1+\frac{\frac{t_3}{t_3-t_2}-1}{1-f_1}\leq\frac{t_3}{t_3-t_2} \qquad \text{and} \qquad 1+\frac{\frac{2t_1t_3+t_3-t_1}{1+t_1t_3}-1}{1-f_1-f_2}\leq 2(\frac{2t_1t_3+t_3-t_1}{1+t_1t_3})-1\ .$$

We also have the conditions $t_2<1$ and $t_1>0$. $t_2<1$ implies $\frac{t_3}{t_3-t_2}\leq\frac{t_3}{t_3-1}$. Moreover

$$2\left(\frac{2t_1t_3+t_3-t_1}{1+t_1t_3}\right)-1=2\left(2+\frac{t_3-t_1-2}{1+t_1t_3}\right)-1$$

decreases in value as t_1 increases. Its maximum value is less than the value for $t_1=0$, because of the condition $t_1>0$. It follows that $2(\frac{2t_1t_3+t_3-t_1}{1+t_1t_3})-1\leq 2t_3-1$. After putting these relaxations into the constraints of LP (5.17), we get

$$\begin{array}{lll} \min & s_1+s_2+s_3 & \\ & \dfrac{t_3}{t_3-1}s_1+s_2+s_3 & \geq 1 \quad \text{(Triangle } T_1\text{)} \\ & s_1+(2t_3-1)s_2+s_3 & \geq 1 \quad \text{(Triangle } T_2\text{)} \\ & s\in\mathbb{R}^3_+. & \end{array} \tag{5.18}$$

The optimal solution of LP (5.18) is

$$s_3=0,\quad s_1=\frac{2(t_3-1)^2}{2t_3^2-2t_3+1},\quad s_2=\frac{1}{2t_3^2-2t_3+1},\quad \text{and}\quad s_1+s_2+s_3=\frac{2t_3^2-4t_3+3}{2t_3^2-2t_3+1}\ .$$

Under the condition $t_3>1$, the minimum value of $s_1+s_2+s_3$ is achieved for $t_3=1+\sqrt{\frac{1}{2}}$ with value $\frac{1}{1+\sqrt{\frac{1}{2}}}>\frac{1}{2}$. □

5.6 Split closure vs. a single triangle or quadrilateral inequality

In this section, we prove Theorem 69.

This is done by showing that there exist examples of integer programs (5.1) where the optimal value for optimizing in the direction of a triangle (or quadrilateral) inequality over the split closure S_f^k can be arbitrarily small. We give such examples for facets derived from triangles of Type 2 and Type 3, and from quadrilaterals.

These examples have the property that the point f lies in the relative interior of a segment joining two integral points at distance 1.

Furthermore, in these examples, the rays end on the boundary of the triangle or quadrilateral and hence the facet corresponding to it is of the form $\sum_{j=1}^k s_j \geq 1$. We show that the following LP has optimal value much less than 1.

$$\begin{aligned} z_{SPLIT} = \quad \min \quad & \sum_{j=1}^k s_j \\ & \sum_{j=1}^k \psi(r^j) s_j \quad \geq 1 \quad \text{for all splits } B_\psi \\ & s \in \mathbb{R}_+^k. \end{aligned} \tag{5.19}$$

Theorem 64 then implies Theorem 69.

A key step in the proof is a method for constructing a polyhedron contained in the split closure (Lemma 89). The resulting LP implies an upper bound on z_{SPLIT}. We then give a family of examples showing that this upper bound can be arbitrarily close to 0. We start the proof with an easy lemma.

5.6.1 An easy lemma

Refer to Figure 5.10 for an illustration of the following lemma.

Lemma 87. *Let r^1 and r^2 be two rays that are not multiples of each others and let H_1 and H_2 be the half-lines generated by nonnegative multiples of r^1 and r^2 respectively. Let $p := k_1\, r^1 + k_2\, r^2$ with $k_1, k_2 > 0$. Let L_1, L_2, and L_3 be three distinct lines going*

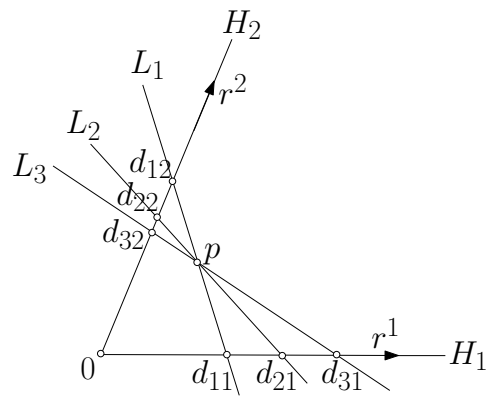


Figure 5.10: Illustration for Lemma 87

through p such that each of the lines intersects both H_1 and H_2 at points other than the origin. Let d_{ij} be the distance from the origin to the intersection of line L_i with the half-line H_j for $i = 1, 2, 3$ and $j = 1, 2$. Assume that $d_{11} < d_{21} < d_{31}$. Then there exists $0 < \lambda < 1$ such that

$$\frac{1}{d_{21}} = \lambda \frac{1}{d_{11}} + (1-\lambda) \frac{1}{d_{31}} \qquad \textit{and} \qquad \frac{1}{d_{22}} = \lambda \frac{1}{d_{12}} + (1-\lambda) \frac{1}{d_{32}}.$$

Proof. Let u^i be a unit vector in the direction of r^i for $i = 1, 2$. Using $\{u^1, u^2\}$ as a base of $\mathbb{R}^2$, for $i = 1, 2, 3$, L_i has equation

$$\frac{1}{d_{i1}} x_1 + \frac{1}{d_{i2}} x_2 = 1$$

As L_2 is a convex combination of L_1 and L_3, there exists $0 < \lambda < 1$ such that $\lambda L_1 + (1-\lambda) L_3 = L_2$. The result follows. □

Corollary 88. *In the situation of Lemma 87, let L_4 be a line parallel to r^1 going through p. Let d_{42} be the distance between the origin and the intersection of H_2 with L_4. Then there exists $0 < \lambda < 1$ such that*

$$\frac{1}{d_{21}} = \lambda \frac{1}{d_{11}} \qquad \textit{and} \qquad \frac{1}{d_{22}} = \lambda \frac{1}{d_{12}} + (1-\lambda) \frac{1}{d_{42}}.$$

Proof. Similar to the proof of Lemma 87. □

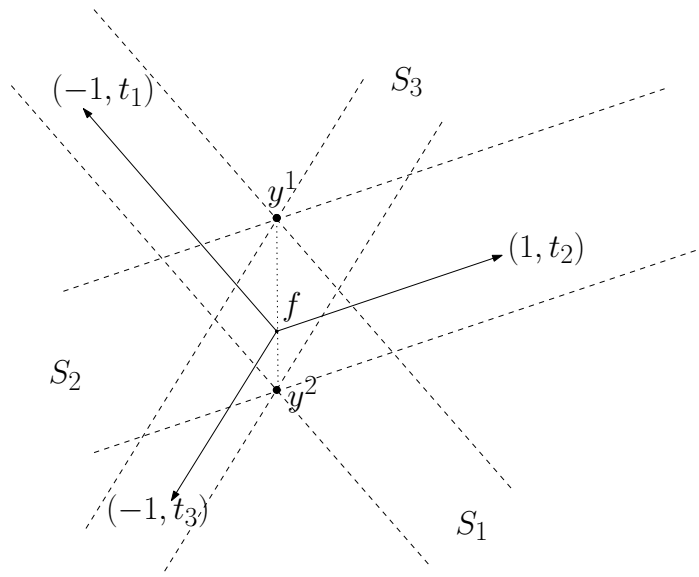


Figure 5.11: Dominating the Split closure with pseudo-splits

5.6.2 A polyhedron contained in the split closure

Our examples for proving Theorem 69 have the property that the point f lies in the relative interior of a segment joining two integral points y^1, y^2 at distance 1.

To obtain an upper bound on the value z_{SPLIT} of the split closure, we define some inequalities which dominate the split closure (5.19). A *pseudo-split* is the convex set between two distinct parallel lines passing through y^1 and y^2 respectively. The direction of the lines, called *direction* of the pseudo-split, is a parameter. Figure 5.11 illustrates three pseudo-splits in the directions of three rays. The *pseudo-split inequality* is derived from a pseudo-split exactly in the same way as from any maximal lattice-free convex set. Note that pseudo-splits are in general not lattice-free and hence do not generate valid inequalities for R_f^k. However, we can dominate any split inequality cutting f by an inequality derived from these convex sets. Indeed, consider any split S containing the fractional point f in its interior. Since f lies on the segment y^1y^2, both boundary lines of S pass through the segment y^1y^2. The pseudo-split with direction identical to the direction of S generates an inequality that dominates the split inequality derived from S, as the coefficient for any ray is smaller in the pseudo-split inequality.

The next lemma states that we can dominate the split closure by using only

the inequalities generated by the pseudo-splits with direction parallel to the rays $r^1, \ldots, r^k$ under mild assumptions on the rays.

Lemma 89. *Assume that none of the rays $r^1, \ldots, r^k$ has a zero first component and that $f = (0, f_2)$ with $0 < f_2 < 1$. Let $y^1 = (0, 1)$ and $y^2 = (0, 0)$, these two points being used to construct pseudo-splits. Let $S_1, \ldots, S_k$ be the pseudo-splits in the directions of rays $r^1, \ldots, r^k$ and denote the corresponding minimal functions by $\psi_{S_1}, \ldots, \psi_{S_k}$. Let S be any split with f in its interior and let S' be the corresponding pseudo-split. Then the inequality $\sum_{j=1}^{k} \psi_{S'}(r^j)s_j \geq 1$ corresponding to S' is dominated by a convex combination of the inequalities $\sum_{j=1}^{k} \psi_{S_i}(r^j)s_j \geq 1$, $i = 1, \ldots, k$. Therefore, the split inequality corresponding to S is dominated by a convex combination of the inequalities corresponding to $\psi_{S_1}, \ldots, \psi_{S_k}$.*

Proof. As a convention, the direction of a pseudo-split forms an angle with the x_1-axis in the range of $]-\frac{\pi}{2}, \frac{\pi}{2}[$. Without loss of generality, assume that the slope of the directions of the pseudo-splits corresponding to the rays $r^1, \ldots r^k$ are monotonically non increasing. We can assume that the direction of S' is different than the direction of any of the rays in $\{r^1, \ldots, r^k\}$ as otherwise the result trivially holds.

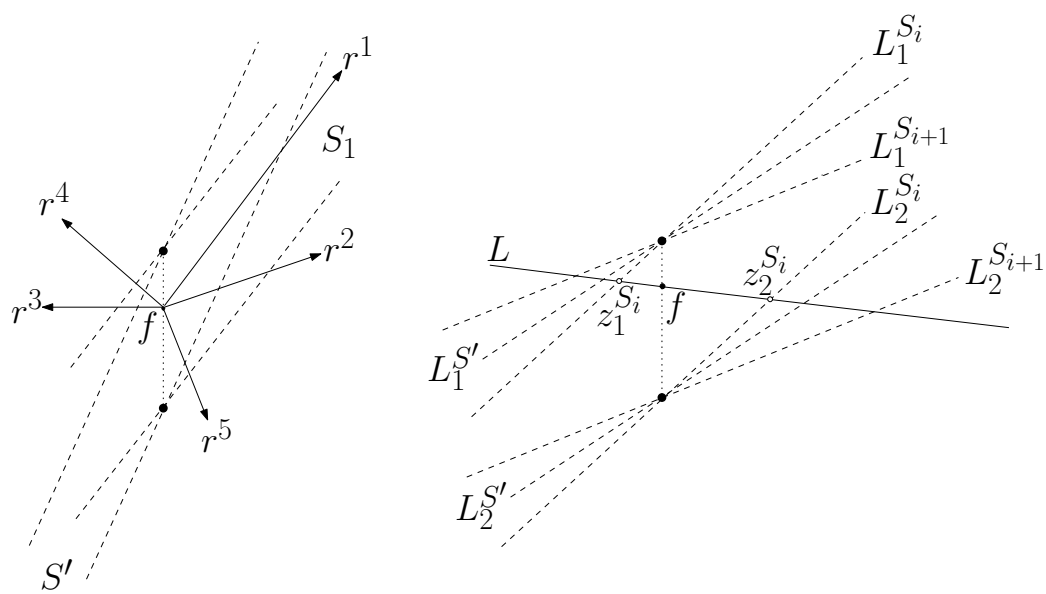


Figure 5.12: Bounding the split closure with a finite number of pseudo-splits

First note that, if S' has a direction with slope greater than the slope of r^1, then the inequality generated by S' is dominated by the one generated by S_1. Indeed,

any ray r^j having a slope smaller than r^1 has its boundary point for S' closer to f than the one for S_1. It follows that $\psi_{S'}(r^j) \geq \psi_{S_1}(r^j)$. See Figure 5.12. A similar reasoning holds for the case where S' has a direction with slope smaller than the slope of r^k.

Thus we only have to consider the case where the slope of the direction of S' is strictly between the slopes of the directions of S_i and S_{i+1}, for some $1 \leq i \leq k-1$. We claim the following.

Observation 1. *There exists a* $0 < \lambda < 1$ *such that* $\psi_{S'}(r) = \lambda\psi_{S_i}(r) + (1-\lambda)\psi_{S_{i+1}}(r)$ *for every ray* $r \in \{r^1, \ldots, r^k\}$.

Proof. For each pseudo-split $S \in \{S', S_i, S_{i+1}\}$, we denote by L_1^S its boundary line passing though $(0,1)$ and by L_2^S its boundary line passing through $(0,0)$.

Consider first any ray r^j with $j < i$ and let L^{r^j} be the half-line $f + \mu r^j$, $\mu \geq 0$. We have that L^{r^j} has a slope greater than the slope of the direction of S_i and thus L^{r^j} intersects the boundaries of S', S_i and S_{i+1} on $L_1^{S'}$, $L_1^{S_{i+1}}$ and $L_1^{S_i}$. By Lemma 87, there exists a $0 < \lambda_1 < 1$ such that, for all $r \in \{r^1, \ldots, r^{i-1}\}$

$$\psi_{S'}(r) = \lambda_1\psi_{S_i}(r) + (1-\lambda_1)\psi_{S_{i+1}}(r) \ . \tag{5.20}$$

By Corollary 88, equation (5.20) also holds for $r = r^i$.

Using a similar reasoning for the rays $\{r^{i+1}, \ldots, r^k\}$ and the boundary lines $L_2^{S'}$, $L_2^{S_{i+1}}$ and $L_2^{S_i}$, there exists a $0 < \lambda_2 < 1$ such that, for all $r \in \{r^{i+1}, \ldots, r^k\}$

$$\psi_{S'}(r) = \lambda_2\psi_{S_i}(r) + (1-\lambda_2)\psi_{S_{i+1}}(r) \ . \tag{5.21}$$

It remains to show that $\lambda_1 = \lambda_2$. Consider any line L through f that is not collinear with r^i or r^{i+1} and not going through y^1 and y^2. For each $S \in \{S', S_i, S_{i+1}\}$, let z_1^S (resp. z_2^S) be the intersection of L with L_1^S (resp. L_2^S) and let d_1^S (resp. d_2^S) be the distance from f to z_1^S (resp. z_2^S). See Figure 5.12. By Lemma 87

$$\frac{1}{d_1^{S'}} = \lambda_1 \frac{1}{d_1^{S_i}} + (1-\lambda_1)\frac{1}{d_1^{S_{i+1}}} \qquad \text{and} \qquad \frac{1}{d_2^{S'}} = \lambda_2 \frac{1}{d_2^{S_i}} + (1-\lambda_2)\frac{1}{d_2^{S_{i+1}}}. \tag{5.22}$$

The length of segments fy^1, fz_1^S, fy^2 and fz_2^S are respectively $1-f_2, d_1^S, f_2$ and d_2^S. Observe that the triangles $fz_1^Sy^1$ and $fz_2^Sy^2$ are homothetic with homothetic ratio $t = \frac{1-f_2}{f_2}$. It follows that $\frac{d_1^S}{d_2^S} = t$. Substituting d_1^S by $t \cdot d_2^S$ in (5.22) yields $\lambda_1 = \lambda_2$. $\square$

This observation proves the lemma.

□

Using the above lemma, we can bound the split closure for three rays. We assume that none of the rays has a zero first component and that the three rays generate $\mathbb{R}^2$. Without loss of generality, we make the following assumptions. The rays are $r^1 = \mu_1(-1, t_1)$, $r^2 = \mu_2(1, t_2)$ and $r^3 = \mu_3(-1, t_3)$, where t_i's are rational numbers in the range $]-\infty, \infty[$, with $t_1 > t_3$ and μ_i's are scaling factors with $\mu_i > 0$. Any configuration of three rays satisfying the above assumptions either fits this description or is a reflection of it about the segment $(0, 0), (0, 1)$. In addition, we must have $-t_1 < t_2 < -t_3$. See Figure 5.11 for an illustration.

Theorem 90. *Assume that $f = (0, f_2)$ with $0 < f_2 < 1$. Consider rays $r^1 = \mu_1(-1, t_1)$, $r^2 = \mu_2(1, t_2)$ and $r^3 = \mu_3(-1, t_3)$, where t_i's are rational numbers with $-t_1 < t_2 < -t_3$ and $\mu_i > 0$. Then*

$$z_{SPLIT} \leq \frac{1}{t_1 - t_3}\left(\frac{1 - f_2}{\mu_1} + \frac{f_2}{\mu_3}\right) .$$

Proof. Let $y^1 = (0, 1)$ and $y^2 = (0, 0)$, these two points being used to construct pseudo-splits. By Lemma 89, we know that the three pseudo-splits S_1, S_2, S_3 corresponding to the directions of r^1, r^2, r^3 dominate the entire split closure. More formally, the following LP is a strengthening of (5.19) in this example of three rays.

$$\begin{array}{ll} \min & s_1 + s_2 + s_3 \\ & \psi_{S_1}(r^1)s_1 + \psi_{S_1}(r^2)s_2 + \psi_{S_1}(r^3)s_3 \geq 1 \\ & \psi_{S_2}(r^1)s_1 + \psi_{S_2}(r^2)s_2 + \psi_{S_2}(r^3)s_3 \geq 1 \\ & \psi_{S_3}(r^1)s_1 + \psi_{S_3}(r^2)s_2 + \psi_{S_3}(r^3)s_3 \geq 1 \\ & s \in \mathbb{R}^3_+. \end{array} \tag{5.23}$$

It is fairly straightforward to compute the coefficients in the above inequalities. We give the calculations for S_1; the coefficients for the other two follow along similar lines.

$\psi_{S_1}(r^1)$ is 0, since r^1 is parallel to the direction of S_1.

Consider r^2 and let its boundary point p for S_1 be $(0, f_2) + \gamma\mu_2(1, t_2)$, for some $\gamma \geq 0$. Then $\psi_{S_1}(r^2)$ is $\frac{1}{\gamma}$. To compute γ, we observe that p is on boundary 1 of S_1,

by assumption of $t_2 > -t_1$. Hence, the slope of the line connecting p and $(0,1)$ is $-t_1$. Therefore,

$$\frac{f_2 + \gamma\mu_2 t_2 - 1}{0 + \gamma\mu_2} = -t_1$$

which yields $\gamma = \frac{1-f_2}{\mu_2(t_1+t_2)}$. Hence $\psi_{S_1}(r^2) = \frac{\mu_2(t_1+t_2)}{1-f_2}$.

Now consider r^3. As before, let its boundary point p' for S_1 be $(0, f_2)+\gamma'\mu_3(-1, t_3)$, for some $\gamma' \geq 0$. This time note that the ray intersects boundary 2 (by the assumption $t_3 < t_1$). Equating slopes, we get

$$\frac{f_2 + \gamma'\mu_3 t_3}{0 - \gamma'\mu_3} = -t_1$$

which yields $\gamma' = \frac{f_2}{\mu_3(t_1-t_3)}$. Hence $\psi_{S_1}(r^3) = \frac{\mu_3(t_1-t_3)}{f_2}$.

So we have that the inequality corresponding to S_1 is

$$0 \cdot s_1 + \frac{\mu_2(t_1+t_2)}{1-f_2}s_2 + \frac{\mu_3(t_1-t_3)}{f_2}s_3 \geq 1 \ .$$

By very similar calculations, we can get the inequalities corresponding to ψ_{S_2} and ψ_{S_3}. LP (5.23) becomes

$$\begin{array}{lll} \min & s_1 + s_2 + s_3 & \\ & 0 \cdot s_1 + \dfrac{\mu_2(t_1+t_2)}{1-f_2}s_2 + \dfrac{\mu_3(t_1-t_3)}{f_2}s_3 & \geq 1 \\ & \dfrac{\mu_1(t_1+t_2)}{1-f_2}s_1 + 0 \cdot s_2 + \dfrac{\mu_3(-t_3-t_2)}{f_2}s_3 & \geq 1 \\ & \dfrac{\mu_1(t_1-t_3)}{1-f_2}s_1 + \dfrac{\mu_2(-t_3-t_2)}{f_2}s_2 + 0 \cdot s_3 & \geq 1 \\ & s \in \mathbb{R}^3_+. & \end{array} \tag{5.24}$$

As a sanity check, note that the assumption $-t_1 < t_2 < -t_3$ implies that all the coefficients are nonnegative.

The following solution is feasible for LP (5.24):

$$s_1 = \frac{1-f_2}{\mu_1(t_1-t_3)}, \quad s_2 = 0, \quad s_3 = \frac{f_2}{\mu_3(t_1-t_3)} \quad \text{and} \quad s_1+s_2+s_3 = \frac{1}{t_1-t_3}\left(\frac{1-f_2}{\mu_1} + \frac{f_2}{\mu_3}\right) \ .$$

Since the above LP was a strengthening of (5.19), we obtain

$$z_{SPLIT} \leq s_1 + s_2 + s_3 = \frac{1}{t_1 - t_3}\left(\frac{1-f_2}{\mu_1} + \frac{f_2}{\mu_3}\right) .$$

□

If the rays are such that $\mu_1 = \mu_3 = 1$, then the above expression is $\frac{1}{t_1 - t_3}$. This implies that in this case if we have rays such that $(t_1 - t_3)$ tends to infinity, then z_{SPLIT} tends to 0.

A slightly more general case

We now generalize slightly the previous case. We now let the fractional point $f = (f_1, f_2)$ be off the segment between two lattice points at distance 1. However, the critical assumption we make is that two of the three rays pass through that segment. See Figure 5.13.

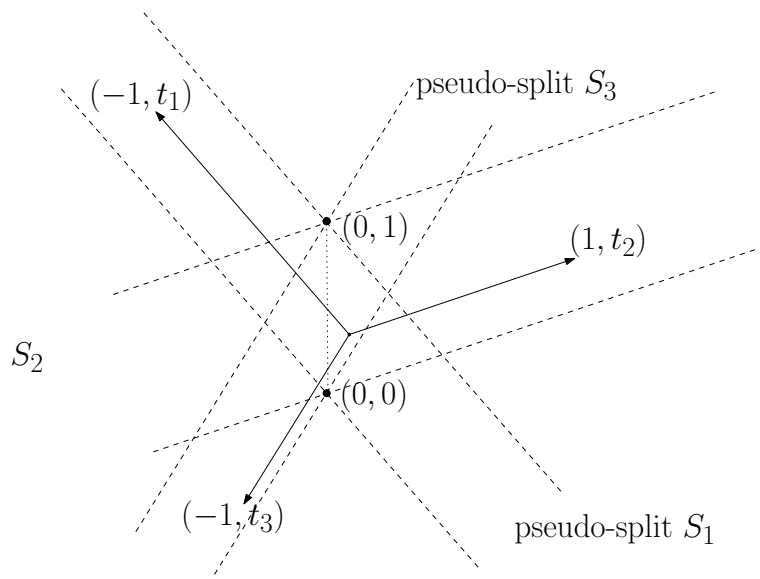


Figure 5.13: Bounding the Split Closure : Case 2

First we need a result similar to Lemma 89 so to formulate a relatively small LP to bound z_{SPLIT}. To get that result, the assumptions on the position of the fractional point and the directions of the two rays are crucial.

Lemma 91. *Consider a fractional point f and three rays r^1, r^2, r^3. If two of the rays point to the interior of an edge connecting two neighboring lattice points y^1, y^2, then any pseudo-split passing through y^1 and y^2 is dominated by the convex combination of the three pseudo-splits parallel to r^1, r^2, r^3.*

Proof. Consider any other pseudo-split S'. Suppose that the slope of the direction of S' lies between those of S_1 and S_2. We know from Observation 1 that if f is on the segment connecting $(0,0)$ and $(0,1)$, then there exists $0 < \lambda_1 < 1$ such that

$$\psi_{S'}(r^j) = \lambda_1 \psi_{S_1}(r^j) + (1-\lambda_1)\psi_{S_2}(r^j) \text{ for } j = 1,2,3. \tag{5.25}$$

Let us also assume, without loss of generality, that r^1 and r^2 intersect the boundary line L_1^i passing through $(0,1)$) of the pseudo-splits S_i for $i = 1,2,3$. Suppose that f is moved along a parallel to the direction of S' in such a way that r^1 and r^3 pass through the segment connecting $(0,0)$ and $(0,1)$. Corollary 88 says that there exists $0 < \lambda_2 < 1$ such that $\psi_{S'}(r^j) = \lambda' \psi_{S_1}(r^j) + (1-\lambda')\psi_{S_2}(r^j)$ for $j = 1,2$.

Claim 17. $\lambda_2 \psi_{S_1}(r^3) + (1-\lambda_2)\psi_{S_2}(r^3) \leq \psi S'(r^3)$*, where λ_2 is as defined above.*

This implies that S' is dominated by a convex combination of S_1 and S_2.

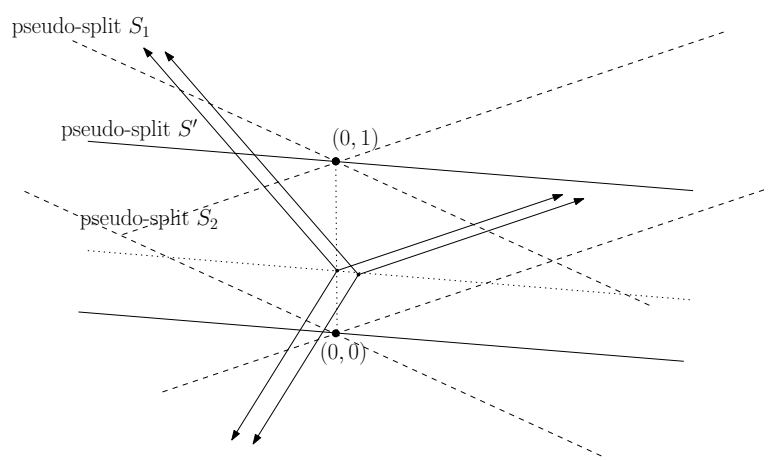


Figure 5.14: Dominating any split in the general case

To prove Claim 17, let us first compute λ_1 from Equation (5.25). Let the distances between f and the boundary points of ray r^1 for the pseudo-splits S', S_1 and

S_2 be denoted by $d_{S'}$, d_{S_1} and d_{S_2} respectively. See Figure 5.14. Then using the equation

$$\lambda_1 \frac{1}{d_{S_1}} + (1-\lambda_1)\frac{1}{d_{S_2}} = \frac{1}{d_{S'}} \ ,$$

we get that $\lambda_1 = \frac{(d_{S_2}-d_{S'})d_{S_1}}{(d_{S_2}-d_{S_1})d_{S'}}$.

If we move the fractional point along a parallel to the direction of split S', we get that λ_2 as in Claim 17 is equal to $\frac{(d_{S_2}-d_{S'})d'_{S_1}}{(d_{S_2}-d_{S_1})d_{S'}}$, where d'_{S_1} is the distance cut by the new ray on boundary 1 of S_1. See Figure 5.14.

It follows that $\lambda_2 \leq \lambda_1$ as we move the point in the manner described, since $d'_{S_1} \leq d_{S_1}$.

For ray r^3, which intersects boundary 2 of the three pseudo-splits, define λ' such that $\psi_{S'}(r^3) = \lambda'\psi_{S_1}(r^3) + (1-\lambda')\psi_{S_2}$. Recall from the Equation (5.25) that when f was on the segment, $\psi_{S'}(r^3) = \lambda_1\psi_{S_1}(r^3)+(1-\lambda_2)\psi_{S_2}$. Observe that this now implies $\lambda_1 \leq \lambda'$ (since the role of S_1 and S_2 are reversed for r^3), which implies $\lambda_2 \leq \lambda'$. Also note that $\psi_{S_1}(r^3) \geq \psi_{S_2}(r^3)$. Therefore, $\psi_{S'}(r^3) \geq \lambda_2\psi_{S_1}(r^3) + (1-\lambda_2)\psi_{S_2}$. this proves Claim 17.

□

Note that the point f is in the square {(0,0),(1,0),(1,1),(0,1)}. This implies that any split which contains f, but does not pass through the segment $(0,0),(0,1)$, is dominated by pseudo-split S_2. So to dominate the split closure in this case, we only need to consider the inequalities derived from the pseudo-splits S_1, S_2, S_3.

We can derive the constraints corresponding to S_1, S_2, S_3 similarly to the previous section. We have then to find an upper bound on the value of the following LP.

$$\begin{array}{lll} \min & s_1 + s_2 + s_3 & \\ & 0 \cdot s_1 + \dfrac{\mu_2(t_1+t_2)}{1-f_2-f_1t_1}s_2 + \dfrac{\mu_3(t_1-t_3)}{f_2+f_1t_1}s_3 & \geq 1 \\ & \dfrac{\mu_1(t_1+t_2)}{1-f_2+f_1t_2}s_1 + 0 \cdot s_2 + \dfrac{\mu_3(-t_3-t_2)}{f_2-f_1t_2}s_3 & \geq 1 \\ & \dfrac{\mu_1(t_1-t_3)}{1-f_2-f_1t_3}s_1 + \dfrac{\mu_2(-t_3-t_2)}{f_2+f_1t_3}s_2 + 0 \cdot s_3 & \geq 1 \\ & s \in \mathbb{R}^3_+. & \end{array} \tag{5.26}$$

The upper bound can be obtained by exhibiting a feasible solution. Recall that we had made the assumption that r^1, r^2, r^3 span $\mathbb{R}^2$, which implies $-t_1 \leq t_2 \leq -t_3$. Under this assumption, one feasible solution is

$$s_1 = \frac{1 - f_2 - f_1 t_3}{\mu_1(t_1 - t_3)}, \quad s_2 = 0 \text{ and } s_3 = \frac{f_2 + f_1 t_1}{\mu_3(t_1 - t_3)} .$$

It clearly satisfies the first and third constraints in LP (5.26). If we substitute s_1 and s_3 into the second constraint, the left hand side is

$$\begin{aligned}
&\frac{t_1 + t_2}{1 - f_2 + f_1 t_2} \cdot \frac{1 - f_2 - f_1 t_3}{t_1 - t_3} + \frac{-t_3 - t_2}{f_2 - f_1 t_2} \cdot \frac{f_2 + f_1 t_1}{t_1 - t_3} \\
&= \frac{t_1 + t_2}{t_1 - t_3} \cdot \frac{1 - f_2 - f_1 t_3}{1 - f_2 + f_1 t_2} + \frac{-t_3 - t_2}{t_1 - t_3} \cdot \frac{f_2 + f_1 t_1}{f_2 - f_1 t_2} \\
&\geq \frac{t_1 + t_2}{t_1 - t_3} + \frac{-t_3 - t_2}{t_1 - t_3} = 1 .
\end{aligned}$$

The first inequality follows from the assumptions $-t_1 \leq t_2 \leq -t_3$. The value of this feasible solution is

$$s_1 + s_2 + s_3 = \frac{1}{t_1 - t_3} \left(\frac{1 - f_2 - f_1 t_3}{\mu_1} + \frac{f_2 + f_1 t_1}{\mu_3} \right) .$$

If the rays are such that $\mu_1 = \mu_3 = 1$, then the objective value becomes $f_1 + \frac{1}{t_1 - t_3}$. Also, by the assumption that r^1 and r^3 pass through the edge connecting $(0, 0)$ and $(0, 1)$, $f_1 \leq \frac{1}{t_1 - t_3}$. So the value of the LP is bounded by $\frac{2}{t_1 - t_3}$ if $\mu_1 = \mu_3 = 1$. Like the previous section, this implies that as $(t_1, -t_3)$ tends to infinity, z_{SPLIT} tends to 0.

5.6.3 Type 2 triangles that do much better than the split closure

In Section 5.6.2, we showed that we can bound the value of the split closure under mild conditions on f and the rays. In particular, we showed that as $t_1 - t_3$ increases in value, the split closure does arbitrarily bad. In this section, we consider an infinite family of Type 2 triangles with rays pointing to its corners which satisfy these conditions.

Consider the same situation as in Section 5.6.2 and consider the Type 2 triangle T with the following three edges. The line parallel to the x_2-axis and passing through $(-1, 0)$ supports one of the edges, and the other two edges are supported

by lines passing through $(0,1)$ and $(0,0)$ respectively. See left part of Figure 5.15. Note that in this example, the rays are of the form $r^1 = (-1, t_1), r^2 = \mu(1, t_2), r^3 = (-1, t_3)$. In the notation of Section 5.6.2, $\mu_1 = \mu_3 = 1$.

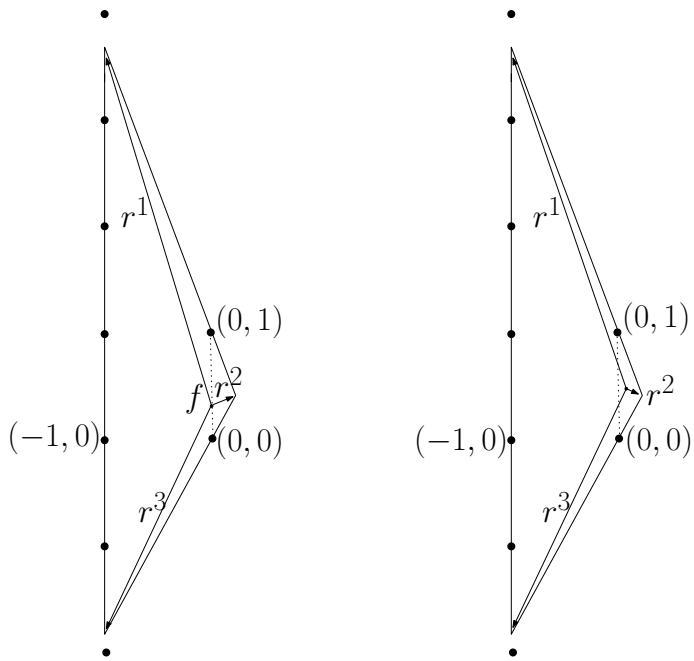


Figure 5.15: Facets from Type 2 triangles with large gap versus the split closure

Let Δ be the triangle with vertices $(0,0)$, $(0,1)$ and the vertex x^2 of T with positive first coordinate (see right part of Figure 5.15).

Theorem 92. *Given any large constant $\alpha > 1$, there exists a Type 2 triangle T as shown in Figure 5.15 such that for any point f in the interior of Δ or in the relative interior of the segment joining $(0,0)$ to $(0,1)$, the LP (5.19) has value $z_{SPLIT} \leq \frac{1}{\alpha}$.*

Proof. Let $M = \lceil \alpha \rceil$. When the fractional point f is on the segment connecting $(0,0)$ and $(0,1)$, consider the triangle T with M integral points in the interior of the vertical edge (the triangle on the left in Figure 5.15). This implies $t_1 - t_3 \geq M$. Therefore, from the result of Section 5.6.2, $\mu_1 = \mu_3 = 1$ implies that $z_{SPLIT} \leq \frac{1}{t_1 - t_3} \leq \frac{1}{\alpha}$.

When the fractional point is not on the segment connecting $(0,0), (0,1)$ but is in the interior of triangle Δ with vertices $(0,0)$, $(0,1)$ and the vertex x^2 of T with

positive first coordinate (on the right in Figure 5.15), consider a triangle T with $2M$ integral points on the vertical edge. Thus, $t_1 - t_3 \geq 2M$. From Section 5.6.2, $\mu_1 = \mu_3 = 1$ implies that $z_{SPLIT} \leq \frac{2}{t_1+t_3} \leq \frac{1}{M} \leq \frac{1}{\alpha}$. □

In these examples, for any large constant α, optimizing over the split closure in the direction of the facet defined by these Type 2 triangles yields at most $\frac{1}{\alpha}$. This implies Theorem 69.

However, such bad examples cannot be constructed for any position of point f in the triangle T. In particular, define the triangle Δ' obtained from Δ by a homothetic transformation with center x^2 and factor 2 (so one vertex of Δ' is x^2 and points $(0,0)$ and $(0,1)$ become the middle points of the two edges of Δ' with endpoint x^2). When f is an interior point of T outside Δ', it is easy to see that the split inequality obtained from the split parallel to the x_2-axis $-1 \leq x_1 \leq 0$ approximates the triangle inequality defined by T to within a factor at most 2. Indeed the linear program is

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 \\ & s_1 + \frac{f_1 - u}{f_1} s_2 + s_3 \quad \geq 1 \\ & s \in \mathbb{R}^3_+, \end{aligned} \tag{5.27}$$

where u is the first coordinate of x^2. The optimal solution is $s_1 = 0$, $s_2 = \frac{f_1}{f_1 - u}$, $s_3 = 0$. Thus $s_1 + s_2 + s_3 = \frac{f_1}{f_1 - u} \geq \frac{1}{2}$ since $f_1 \leq -u$ for any $f \in T \setminus \Delta'$. This implies that the split inequality approximates the triangle inequality by a factor at most 2 when f is outside Δ'.

Triangles of Type 3 and quadrilaterals

We now show how to modify the construction of Section 5.6.3 to get examples of Type 3 triangles and quadrilaterals that do arbitrarily better than the split closure.

To get a Type 3 triangle, we tilt the vertical edge of the triangle in Figure 5.15 around its integral point with minimum x_2-value. See Figure 5.16. The same bound on z_{SPLIT} is then achieved.

Similarly, quadrilaterals can be constructed by breaking the vertical edge in Figure 5.15 into two edges of the quadrilateral. See Figure 5.16. By very similar arguments as in the previous section, we can show that z_{SPLIT} tends to 0.

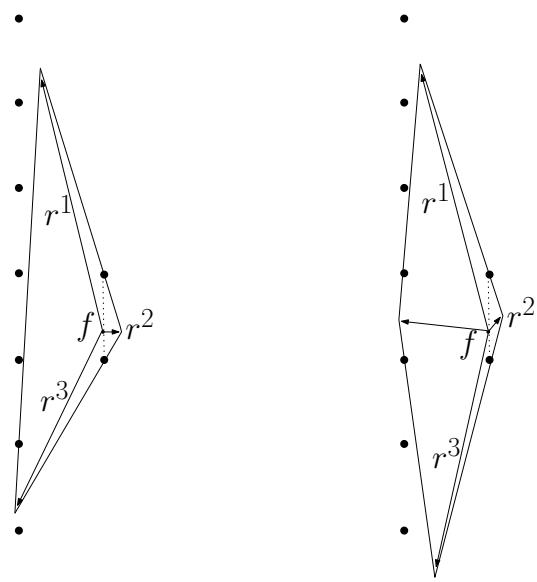


Figure 5.16: Facets from Type 3 triangles and quadrilaterals on which the split closure does poorly

Chapter 6

Experiments with Two Constraint Cuts

In Chapter 5, we provided examples of integer programs with two constraints and two integer variables, where the integrality gap can be closed by a single inequality derived from the two constraints, while the value of the linear relaxation obtained by adding all split cuts is arbitrarily close to 0. These examples suggest that certain cuts derived from two equations might improve the efficiency of MILP solvers. In this chapter, we test the empirical effectiveness of this class of two-constraint cuts. The results presented are based on joint work with Pierre Bonami, Gerard Cornuejols and Francois Margot [18].

This chapter is organized as follows. In Section 6.1, describe how we choose the triangle inequalities used in our separation algorithm. In Section 6.2, we give closed form formulas for certain types of cuts, discuss integer lifting of these cuts, as well as their strengthening when the basic variables are nonnegative. These derivations will be based on the results in Chapter 3 (S-free convex sets) and Chapter 4 (lifting integer non-basic variables). In Section 6.3, we present computational results. We show that the conclusions are very sensitive to the experimental setup. We present two different setups which lead to opposite conclusions and argue that the setup based on a standard open-source Gomory cut generator is flawed because it overlooks the issues of reliability versus aggressiveness of the cut generator.

6.1 Type 2 Triangles

As described in Chapter 5, maximal lattice-free triangles can be classified into three types : Type 1, Type 2 and Type 3. The results from Chapter 5, specifically Theorem 69 and its proof, suggest that it might be interesting to generate cuts from Type 2 triangles in addition to the classical Gomory Mixed Integer cuts [55] when solving an MILP.

Unfortunately, it appears difficult to separate over all Type 2 triangle cuts, as it seems to be at least as hard as the NP-hard problem of separating over the split closure [26, 33]. In this paper, we separate a very restricted class of Type 2 triangle cuts. Consider the simplex tableau obtained by solving the LP relaxation of MILP. We have

$$x = f + \sum_{j=1}^{k} r^j s_j$$

where x denotes the basic variables and s the nonbasic. Motivated by the examples used for the proof of Theorem 69, we test the effectiveness of cutting planes from two rows of the simplex tableau

$$\begin{aligned} x_i &= f_i + \sum_{j=1}^{k} r_i^j s_j \\ x_\ell &= f_\ell + \sum_{j=1}^{k} r_\ell^j s_j \end{aligned} \tag{6.1}$$

where x_i and x_ℓ are basic integer variables, f_i is integral and f_ℓ is fractional, i.e. at least one of the basic integral variables has an integral value and at least one has a fractional value. Note that this will typically occur when the basis is degenerate. Without loss of generality we can make a translation so that $f_i = 0$ and $f_\ell \in]0, 1[$. We try to construct two Type 2 triangles, one where the edge that contains at least two integral points is on the line $x_i = -1$, and a symmetrical one where this edge is on the line $x_i = 1$. As these two cases are similar, we focus on the first one in the following discussion. For each ray r^j such that the half-line $f + \lambda r^j$ for $\lambda > 0$ intersects the plane $x_i = -1$, we compute that intersection and its projection onto the (x_i, x_ℓ)-plane. Among these projections, we define $p^2 = (p_i^2, p_\ell^2)$ (resp. $p^3 = (p_i^3, p_\ell^3)$) as the point with largest (resp. smallest) x_ℓ-coordinate, i.e.,

$$p^2 = \left(-1, f_\ell - \min_j \left\{ \frac{r_\ell^j}{r_i^j} \mid r_i^j < 0 \right\}\right) \qquad \text{and} \qquad p^3 = \left(-1, f_\ell - \max_j \left\{ \frac{r_\ell^j}{r_i^j} \mid r_i^j < 0 \right\}\right) .$$

Note that p^2 and p^3 are not defined if $r^i_j \geq 0$ for all $j = 1, \ldots, k$. In this case or when $p^2 = p^3$, we do not generate a cut. Otherwise, we consider three cases:

i) There are at least two integral points in the interior of the segment p^2p^3. We construct a Type 2 triangle with vertices p^1, p^2 and p^3, where p^1 is the intersection of the lines passing through p^2 and $(0, 1)$ and through p^3 and $(0, 0)$.

ii) There is exactly one integral point in the interior of the segment p^2p^3. Let $q^2 = (-1, \lceil p^2_\ell \rceil)$, and $q^3 = (-1, \lfloor p^3_\ell \rfloor)$. If $\lceil p^2_\ell \rceil - p^2_\ell \leq p^3_\ell - \lfloor p^3_\ell \rfloor$, we construct a Type 1 or Type 2 triangle whose vertices are p^1, q^2 and p^3 where p^1 is the intersection of the lines passing through q^2 and $(0, 1)$ and through p^3 and $(0, 0)$. Otherwise, we construct a Type 2 triangle whose vertices are p^1, p^2 and q^3, where p^1 is the intersection of the lines passing through p^2 and $(0, 1)$ and through q^3 and $(0, 0)$.

iii) There is no integral point in the interior of the segment p^2p^3. Let $q^2 = (-1, \lceil p^2_\ell \rceil)$, and $q^3 = (-1, \lfloor p^3_\ell \rfloor)$. We construct a split whose sides are the lines through q^2 and $(0, 1)$ and through q^3 and $(0, 0)$.

Therefore, if the optimal tableau of the LP relaxation of the MILP has m basic integer variables with an integer value and n basic integer variables that are fractional, this algorithm generates up to $2nm$ cuts from the optimal simplex tableau. We use variants of this basic algorithm for generating and testing two-constraint cuts.

6.2 Derivation of the cuts

6.2.1 Formulas of the cut for Type 1 and Type 2 triangles

Consider a lattice-free triangle T of Type 1 or 2 with vertices p^1, p^2 and p^3. Without loss of generality, assume that T is in a canonical form where $p^2 = (-1, 1 + \eta)$ and $p^3 = (-1, -\mu)$ and $0 \leq \eta \leq \mu \leq \eta + 1$. Furthermore we assume that the points $(0, 0)$ and $(0, 1)$ are respectively in the relative interior of edge p^3p^1 and p^2p^1. A Type 1 or Type 2 triangle can always be put into this canonical form by applying a unimodular transformation (see Lemma 11 and Proposition 26 in [37]).

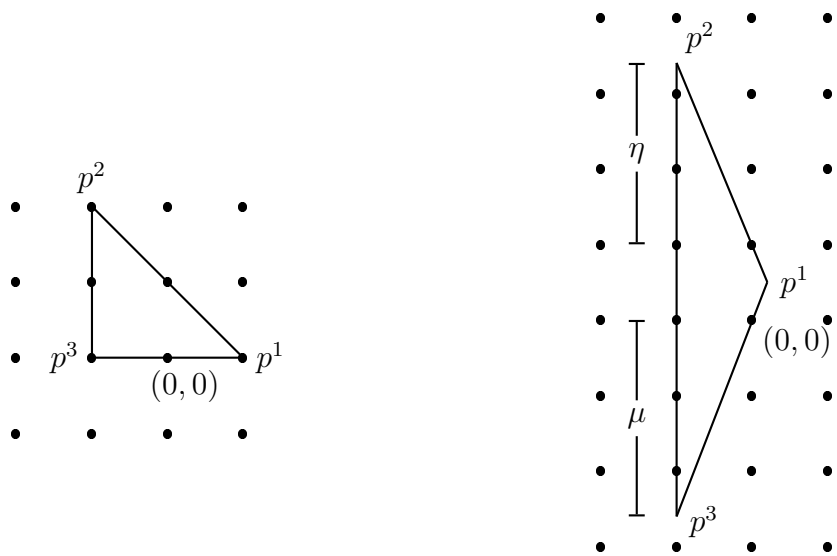


Figure 6.1: Type 1 triangle in canonical form ($\mu = 0$, $\eta = 1$) and Type 2 triangle in canonical form ($0 < \eta \leq \mu \leq \eta + 1$).

For a Type 1 triangle, we thus have $\eta = 0$ and $\mu = 1$ and the three vertices of the triangle are $p^1 = (1, 0)$, $p^2 = (-1, 2)$ and $p^3 = (-1, 0)$, see Figure 6.1. For a Type 2 triangle the points $(-1, 0)$ and $(-1, 1)$ are in the relative interior of the line segment p^2p^3, see Figure 6.1.

The facets of the triangle are given by

$$(-x_i \geq 1) \quad \vee \quad (\eta x_i + x_\ell \geq 1) \quad \vee \quad (\mu x_i - x_\ell \geq 0) \tag{6.2}$$

Using the formula for the minimal valid function associated with such a lattice-free convex set (recall the definition of ψ_B associated with a maximal lattice-free sets B), we have the valid inequality

$$\sum_{j=1}^{k} \max \left\{ \frac{-r_i^j}{1+f_i}, \frac{\eta r_i^j + r_\ell^j}{1-\eta f_i - f_\ell}, \frac{\mu r_i^j - r_\ell^j}{-\mu f_i + f_\ell} \right\} s_j \geq 1 \ . \tag{6.3}$$

In the next two sections, we show how the cut (6.3) can be strengthened if some of the non-basic variables are integer constrained (this operation is called an integer lifting of the cut) or if x_i is nonnegative. These strengthenings are based on ideas presented by Dey and Wolsey [37] and Cornuejols, Conforti and Zambelli [28].

6.2.2 Integer Lifting

In this section, we consider a variation of the two-constraint problem (6.1) where some of the variables s_j are constrained to be integer

$$\begin{aligned} x_i &= f_i + \sum_{j=1}^{k} r_i^j s_j \\ x_\ell &= f_\ell + \sum_{j=1}^{k} r_\ell^j s_j \\ x_i, x_\ell &\in \mathbb{Z}, \\ s_j &\in \mathbb{Z} \quad \text{for all } j \in I \subseteq \{1, \ldots, k\}, \\ s_j &\geq 0 \,. \end{aligned} \tag{6.4}$$

Clearly inequality (6.3) is valid for (6.4). Dey and Wolsey [37] show how to strengthen the coefficients of the non-basic integer variables in (6.3) to obtain a minimal inequality for (6.4). This strengthening is called *integer lifting*. If the lattice-free set used to generate the cut is a triangle of Type 1 or 2, Dey and Wolsey [37] show that there is a unique integer lifting that produces a minimal inequality for (6.4) and it is obtained through the so-called trivial fill-in function. In this section, we derive a closed-form formula for this trivial lifting. We use the terminology of intersection cuts [11]. This also closely mimics the actual implementation in our code.

We first describe the trivial lifting procedure of Dey and Wolsey using the terminology of disjunctive programming. We will then derive the strengthened cut using those formulas.

Proposition 93. *Let $C = \{(x, s) \in \mathbb{Z}^m \times \mathbb{R}^k : x = f + \sum_{j=1}^{k} r^j s_j, s \geq 0\}$. Let I be a subset of $\{1, \ldots, k\}$, $C^I = C \cap \left\{(x, s) \in \mathbb{Z}^m \times \mathbb{R}^k : s_j \in \mathbb{Z}, \forall j \in I\right\}$ be a mixed-integer set and*

$$\Pi = \left(\pi^1 x \geq \pi_0^1\right) \vee \ldots \vee \left(\pi^p x \geq \pi_0^p\right)$$

be a disjunction satisfied by all $x \in \mathbb{Z}^m$. For $j \in I$, let m^j be an integral vector of dimension m. The disjunction

$$\Pi' = \left(\pi^1 \left(x - \sum_{j \in I} m^j s_j\right) \geq \pi_0^1\right) \vee \ldots \vee \left(\pi^p \left(x - \sum_{j \in I} m^j s_j\right) \geq \pi_0^p\right)$$

is valid for C^I.

Proof. Suppose that the disjunction Π' is not valid. Then there exist $(x, s) \in \mathbb{Z}^m \times \mathbb{R}^k$ with $s_j \in \mathbb{Z}$, $\forall j \in I$, such that:

$$\pi^l(x - \sum_{j \in I} m^j s_j) < \pi_0^l \qquad\qquad l = 1, \ldots, p.$$

Let $x' = x - \sum_{j \in I} m^j s_j$. $x' \in \mathbb{Z}^m$ since $x \in \mathbb{Z}^m$, $m^j \in \mathbb{Z}^m$ and $s_j \in \mathbb{Z}$ for all $j \in I$. Then x' violates disjunction Π which contradicts the fact that Π is a valid disjunction. □

Now we consider the intersection cut obtained from a disjunction of the form Π' stated in Proposition 93.

Proposition 94. *Let* $C = \{(x, s) \in \mathbb{Z}^m \times \mathbb{R}^k : x = f + \sum_{j=1}^k r^j s_j, s \geq 0\}$. *Let* $J = \{1, \ldots, k\}$ *and* I *be a subset of* J, $C^I = C \cap \left\{(x, s) \in \mathbb{Z}^m \times \mathbb{R}^k : s_j \in \mathbb{Z}, \forall j \in I\right\}$ *be a mixed-integer set and*

$$\Pi = (\pi^1 x \geq \pi_0^1) \vee \ldots \vee (\pi^p x \geq \pi_0^p)$$

be a disjunction satisfied by all $x \in \mathbb{Z}^m$ *and not satisfied by the point* f. *For all* $m^j \in \mathbb{Z}^m$, $j = 1, \ldots, m$ *the inequality*

$$\sum_{j \in I} \max_{l=1,\ldots,p} \left\{ \frac{\pi^l(r^j - m^j)}{\pi_0^l - \pi^l f} \right\} s_j + \sum_{j \in J \setminus I} \max_{l=1,\ldots,p} \left\{ \frac{\pi^l r^j}{\pi_0^l - \pi^l f} \right\} s_j \geq 1 \tag{6.5}$$

is valid for C^I.

Proof. Since Π is valid, for all $m^j \in \mathbb{Z}^m$, $j = 1, \ldots, m$ the disjunction

$$\Pi' = (\pi^1(x - \sum_{j \in I} m^j s_j) \geq \pi_0^1) \vee \ldots \vee (\pi^k(x - \sum_{j \in I} m^j s_j) \geq \pi_0^p)$$

is valid.

The terms of the disjunction can be rewritten as:

$$\sum_{j \in I} \pi^l(r^j - m^j) s_j + \sum_{j \in J \setminus I} \pi^l r^j s_j \geq \pi_0^l - \pi^l f. \qquad\qquad l = 1 \ldots, p.$$

The point f does not satisfy the disjunction and therefore $\pi_0^l - \pi^l f > 0$, dividing the inequality by $\pi_0^l - \pi^l f$ we obtain

$$\sum_{j \in I} \frac{\pi^l(r^j - m^j)}{\pi_0^l - \pi^l f} s_j + \sum_{j \in J \setminus I} \frac{\pi^l r^j}{\pi_0^l - \pi^l f} s_j \geq 1 \qquad\qquad l = 1 \ldots, p.$$

Each of the above inequalities is valid for one term of the disjunction. By applying, the disjunctive principle we obtain (6.5). □

This concludes the description of the trivial fill-in function. To derive the formula of the cut strengthened by using the above trivial fill-in function, we define for any $j \in \{1, \ldots, k\}$ and any integers m_1 and m_2:

$$\begin{aligned}
\Phi_1^j(m_1) &= \frac{m_1 - r_i^j}{1 + f_i}, \\
\Phi_2^j(m_1, m_2) &= \frac{\eta\left(r_i^j - m_1\right) + r_\ell^j - m_2}{1 - \eta f_i - f_\ell}, \\
\Phi_3^j(m_1, m_2) &= \frac{\mu\left(r_i^j - m_1\right) - r_\ell^j + m_2}{-\mu f_i + f_\ell},
\end{aligned}$$

and

$$\Phi^j(m_1, m_2) = \max\left\{\Phi_1^j(m_1), \Phi_2^j(m_1, m_2), \Phi_3^j(m_1, m_2)\right\}.$$

The strengthened cut is then given by:

$$\sum_{j \in I} \min_{(m_1, m_2) \in \mathbb{Z}^2} \Phi^j(m_1, m_2) s_j + \sum_{j \in \bar{I}} \Phi^j(0, 0) s_j \geq 1 \ ,$$

where $\bar{I} = \{1, \ldots, k\} \setminus I$. The following proposition gives the closed form formula for $\min_{(m_1, m_2) \in \mathbb{Z}^2} \Phi^j(m_1, m_2)$.

Proposition 95. *Let* $\overline{m}_2^j = r_\ell^j + \frac{\left(r_i^j - \lfloor r_i^j \rfloor\right)(-\mu + (\mu+\eta) f_\ell)}{1 - (\mu+\eta) f_i}$, $\overline{m}_1^j = r_i^j - \frac{(1+f_i)\left(r_\ell^j - \lfloor r_\ell^j \rfloor\right)}{1 + \eta - f_\ell}$ *and* $\widehat{m}_1^j = r_i^j - \frac{(1+f_i)(\lceil r_\ell^j \rceil - r_\ell^j)}{\mu + f_\ell}$.

(i)

$$\min_{\substack{m_1, m_2 \in \mathbb{Z} \\ m_1 \leq r_i^j}} \Phi^j(m_1, m_2) = \min\left\{\Phi_2^j\left(\lfloor r_i^j \rfloor, \lfloor \overline{m}_2^j \rfloor\right), \Phi_3^j\left(\lfloor r_i^j \rfloor, \lceil \overline{m}_2^j \rceil\right)\right\};$$

(ii)

$$\min_{\substack{m_1, m_2 \in \mathbb{Z} \\ m_1 > r_{1j}, m_2 \leq r_\ell^j}} \Phi^j(m_1, m_2) = \begin{cases} \min\left\{\Phi_1^j\left(\lceil \overline{m}_1^j \rceil\right), \Phi_2^j\left(\lfloor \overline{m}_1^j \rfloor, \lfloor r_\ell^j \rfloor\right)\right\} & \textit{if } \overline{m}_1^j > r_i^j, \\ \Phi_2^j(\lceil r_i^j \rceil, \lfloor r_\ell^j \rfloor) & \textit{otherwise.} \end{cases};$$

(iii)

$$\min_{\substack{m_1,m_2\in\mathbb{Z}\\ m_1>r_i^j,m_2\geq r_\ell^j}} \Phi(m_1^j,m_2^j) = \begin{cases} \min\left\{\Phi_1^j\left(\lceil \widehat{m}_1^j\rceil\right), \Phi_3^j\left(\lfloor \widehat{m}_1^j\rfloor, \lceil r_\ell^j\rceil\right)\right\} & \textit{if } \widehat{m}_1^j > r_i^j, \\ \Phi_3^j\left(\lceil r_i^j\rceil, \lceil r_\ell^j\rceil\right) & \textit{otherwise.} \end{cases}.$$

Proof. (i) First we show that if $m_1 \leq r_i^j$, then $\Phi^j(m_1,m_2) = \max\left\{\Phi_2^j(m_1,m_2), \Phi_3^j(m_1,m_2)\right\}$. Note that if $m_1 \leq r_i^j$, $\Phi_1^j(m_1) = \frac{m_1-r_i^j}{1+f_1} \leq 0$. On the other hand, since η and μ are nonnegative and either $m_2 - r_\ell^j$ or $r_\ell^j - m_2$ is nonnegative $\Phi^j(m_1,m_2) \geq 0$.
Since η and μ are positive, if $m_1 \leq \lfloor r_i^j\rfloor$:

$$\frac{\eta\left(r_i^j - m_1\right) + r_\ell^j - m_2}{1-\eta f_1 - f_2} \geq \frac{\eta\left(r_i^j - \lfloor r_i^j\rfloor\right) + r_\ell^j - m_2}{1-f_2-\eta f_1}$$

$$\frac{\mu\left(r_i^j - m_1\right) - r_\ell^j + m_2}{f_2-\mu f_1} \geq \frac{\mu\left(r_i^j - \lfloor r_i^j\rfloor\right) - r_\ell^j + m_2}{f_2-\mu f_1}$$

and therefore $\min\{\Phi^j(m_1,m_2) \mid m_1,m_2\in\mathbb{Z}, m_1\leq r_i^j\} = \min\{\Phi^j(\lfloor r_i^j\rfloor, m_2) \mid m_2 \in \mathbb{Z}\}$. This last minimum is attained either in $\Phi^j(\lfloor r_i^j\rfloor, \lceil \overline{m}_2\rceil)$ or $\Phi^j(\lfloor r_i^j\rfloor, \lfloor \overline{m}_2\rfloor)$. Therefore

$$\min\{\Phi^j(m_1,m_2) \mid m_1,m_2\in\mathbb{Z}, m_1\leq r_i^j\} = \min\left\{\Phi_2^j(\lfloor r_i^j\rfloor, \lfloor \overline{m}_2^j\rfloor), \Phi_3^j(\lfloor r_i^j\rfloor, \lceil \overline{m}_2^j\rceil)\right\}.$$

(ii) We now suppose that $m_1 > r_i^j$ and $m_2 \leq r_\ell^j$. Then $\Phi_3^j(m_1,m_2) \leq 0$ and therefore $\Phi^j(m_1,m_2) = \max\left\{\Phi_1^j(m_1), \Phi_2^j(m_1,m_2)\right\}$. Furthermore, we have:

$$\min\{\Phi(m_1,m_2) \mid m_1,m_2\in\mathbb{Z}, m_1\leq r_i^j, m_2\leq r_\ell^j\} = \min\{\Phi(m_1, \lfloor r_\ell^j\rfloor) \mid m_1\in\mathbb{Z}, m_1\leq r_i^j\}.$$

If $\overline{m}_1 < r_i^j$, the minimum is attained either in $\Phi^j(\lfloor \overline{m}_1\rfloor, \lfloor r_l^j\rfloor)$ or $\Phi(\lceil \overline{m}_1\rceil, \lfloor r_l^j\rfloor)$, otherwise the minimum is attained in $\Phi^j(\lceil r_i^j\rceil, \lfloor r_l^j\rfloor)$. The formula follows.

(iii) Similar to the proof of (ii). □

6.2.3 Using nonnegativity of basic variables

In this section, we consider the case where x_i is constrained to be nonnegative, and provide a closed-form formula to strengthen the cuts (6.3). In most problems, this nonnegativity constraint appears and this strengthening can be achieved.

Intuitively, for a triangle as described in Section 6.1, if we have $x_i \geq 0$ then we can remove the vertical side of the triangle and expand it into an unbounded set containing no feasible solution of (5.1). This is similar to the idea behind using S-free convex sets as described in Chapter 3. We call this set a *wedge*.

We consider a Type 1 or 2 triangle in canonical form together with the feasible set, including the nonnegativity on x_i:

$$\begin{aligned} x_i &= f_i + \sum_{j=1}^{k} r_i^j s_j \\ x_\ell &= f_\ell + \sum_{j=1}^{k} r_\ell^j s_j \\ x_i, x_\ell &\in \mathbb{Z}, \\ x_i &\geq 0 \\ s_j &\in \mathbb{Z} \quad \text{for all } \ j \in I \subseteq \{1, \ldots, k\}, \\ s_j &\geq 0\ . \end{aligned} \tag{6.6}$$

The following are the facets of this wedge

$$(\eta x_i + x_\ell \geq 1) \quad \vee \quad (\mu x_i - x_\ell \geq 0)\ . \tag{6.7}$$

The inequality derived from (6.7) is given by:

$$\sum_{j \in J} \max \left\{ \frac{\eta r_i^j + r_\ell^j}{1 - f_\ell - \eta f_i}, \frac{\mu r_i^j - r_\ell^j}{f_\ell - \mu f_i} \right\} s_j \geq 1. \tag{6.8}$$

Note that the inequality (6.8) dominates (6.3). In particular some of the coefficients of the latter cut may be negative. Balas [10] observed the following.

Proposition 96. *Inequality* (6.8) *is valid for the split closure.*

Proof. Let P be the polyhedron obtained by dropping the integrality constraints in (6.6). Let $P_1 = P \cap \{x_\ell \geq 1\}$ and $P_2 = P \cap \{x_\ell \leq 0\}$. Inequality (6.8) is valid for $(P \cap \{\eta x_i + x_\ell \geq 1\}) \cup (P \cap \{\mu x_i - x_\ell \geq 0\})$. Since $P_1 \subseteq P \cap \{\eta x_i + x_\ell \geq 1\}$ and $P_1 \subseteq P \cap \{\mu x_i - x_\ell \geq 0\}$, inequality (6.8) is valid for $P_1 \cup P_2$, which proves that it is a split inequality. □

An optimal integer lifting for this cut can be computed in a similar way to the trivial lifting computed in Section 6.2.2. The following proposition describes how

the lifting can be adapted for the wedge (6.7). The difference with the lifting performed using Proposition 93 is that here the disjunction can be shifted by any integer in the direction of x_ℓ but only by a positive integer in the direction of x_i. See [28] for a proof that this lifting is the unique minimal lifting.

Proposition 97. *For $j \in I$, let $m^j \in \mathbb{Z}^2$ with $m_1^j \geq 0$. The disjunction*

$$\left(\eta\left(x_i + \sum_{j\in I} m_j^1 s_j\right) + x_\ell - \sum_{j\in I} m_2^j s_j \geq 1\right) \quad \vee \quad \left(\mu\left(x_i + \sum_{j\in I} m_j^1 s_j\right) - x_\ell + \sum_{j\in I} m_2^j s_j \geq 0\right) \tag{6.9}$$

is satisfied by all points $((x_i, x_\ell), s) \in \mathbb{Z}^2 \times \mathbb{R}^n$ with $x_i \geq 0$ and $s_j \in \mathbb{Z}_+ \; \forall j \in I$.

Proof. Suppose that $((x_i, x_\ell), s) \in \mathbb{Z}^2 \times \mathbb{R}^n$ with $x_i \geq 0$ and $s_j \in \mathbb{Z} \; \forall j \in I$ does not satisfy (6.9). Let $x_i' = x_i + \sum_{j\in I} m_1^j s_j$ and $x_\ell' = x_\ell - \sum_{j\in I} m_2^j s_j$. Then $x_i' \in \mathbb{Z}_+$ and $x_\ell' \in \mathbb{Z}$ but (x_i', x_ℓ') does not satisfy (6.7). □

Using this proposition, we can derive a formula for a strengthened cut which uses the integrality of the variables s_j for $j \in I$. For $m_1 \in \mathbb{Z}_+$ and $m_2 \in \mathbb{Z}$, we define

$$\Psi^j(m_1, m_2) = \max\left\{\frac{\eta\left(r_i^j + m_1\right) + r_\ell^j - m_2}{1 - f_\ell - \eta f_i}, \frac{\mu\left(r_i^j + m_1\right) + m_2 - r_\ell^j}{f_\ell - \mu f_i}\right\}.$$

The following cut is valid by application of Proposition 97:

$$\sum_{j\in I} \min_{(m_1,m_2)\in\mathbb{Z}_+\times\mathbb{Z}} \Psi^j(m_1, m_2) s_j + \sum_{j\in J\setminus I} \max\left\{\frac{\eta r_i^j + r_\ell^j}{1 - f_\ell - \eta f_i}, \frac{\mu r_i^j - r_\ell^j}{f_\ell - \mu f_i}\right\} s_j \geq 1 \tag{6.10}$$

The next proposition gives a closed form formula for computing the coefficients of this strengthened cut.

Proposition 98.

$$\min_{(m_1,m_2)\in\mathbb{Z}_+\times\mathbb{Z}} \Psi^j(m_1, m_2) = \min\left\{\frac{\eta r_i^j + r_\ell^j - \lfloor \overline{m}_2^j \rfloor}{1 - f_\ell - \eta f_i}, \frac{\mu r_i^j + \lceil \overline{m}_2^j \rceil - r_\ell^j}{f_\ell - \mu f_i}\right\}$$

with $\overline{m}_2^j = r_\ell^j + \frac{r_i^j(-\mu + (\mu+\eta) f_\ell)}{1 - (\mu+\eta) f_i}$

Proof. Since η and μ are positive and $m_1 \geq 0$

$$\frac{\eta\left(r_i^j - m_1\right) + r_\ell^j - m_2}{1 - \eta f_i - f_\ell} \geq \frac{\eta r_i^j + r_\ell^j - m_2}{1 - f_\ell - \eta f_i}$$
$$\frac{\mu\left(r_i^j - m_1\right) - r_\ell^j + m_2}{f_\ell - \mu f_i} \geq \frac{\mu r_i^j - r_\ell^j + m_2}{f_\ell - \mu f_i}$$

and therefore

$$\min_{(m_1,m_2)\in\mathbb{Z}_+\times\mathbb{Z}} \Psi^j(m_1, m_2) = \min_{m_2\in\mathbb{Z}} \Psi^j(0, m_2).$$

The minimum is attained either in $\Psi^j(0, \lfloor \overline{m}_2^j \rfloor)$ or $\Psi^j(0, \lceil \overline{m}_2^j \rceil)$. □

Note that the cuts generated from the liftings of Propositions 95 and 98 are incomparable, i.e. neither is guaranteed to dominate the other. They will be compared in Section 6.3.

6.3 Computational tests

We now present computational results obtained by applying the separation procedure devised in Section 5.6.3. The cut generator is implemented in C++ using the COIN-OR framework [1] with Cbc-2.2.2 and using Clp-1.8.2 for solving linear programs. The machine used for all the experiments in this section is a 64 bit `Monarch Empro 4-Way Tower Server` with four `AMD Opteron 852 2.6GHz` processors, each with eight `DDR-400` SDRAM of 2 `GB` and running `Linux Fedora 11`. The compiler is `gcc version 4.4.0 20090506 (Red Hat 4.4.0-4)`. Results are obtained using only one processor. Test instances for our experiments are the sixty-eight `MIPLIB3_C_V2` instances [2]. These instances are slight modifications of the standard `MIPLIB3` [24] instances for which the validity of a provided feasible solution can be checked in finite precision arithmetic.

The main motivation behind these tests is to determine if triangle cuts are substantially different from traditional Gomory Mixed Integer (GMI) cuts, and if using them in conjunction with GMI cuts might be useful. In other words, we are not suggesting to replace GMI cuts by triangle cuts. Rather, we want to investigate if triangle cuts can improve performance when used in addition to GMI cuts.

6.3.1 Results on gap closed by our cuts

We compare the following four generators. As described in Section 5.6.3, we partition the rows of the optimal simplex tableau corresponding to the basic integer variables into *fractional rows* and *integer rows*.

- `G` : A cut generator which generates one GMI cut for each fractional row.
- `G-2rounds` : A cut generator which applies `G` to generate a first round of GMI cuts, reoptimizes the resulting LP and generates a second round of GMI cuts from the new optimal basic solution.
- `G+Allpairs` : A cut generator which generates one GMI cut for each fractional row and all cuts derived from every pair of fractional and integer rows as explained in Section 5.6.3. We also add cuts derived by removing the vertical side of the triangle whenever possible, as outlined in Section 6.2.3.
- `G+Deepest` : `G+Allpairs` often generates a large number of cuts. To reduce this number, we only add the GMI cuts and the deepest among all possible triangle cuts for each fractional row. Note that `G` generates one cut per fractional row, whereas `G+Deepest` generates two.

Since computers work in finite precision, cut generators sometimes generate invalid cuts. To limit these occurrences, a variety of tolerances and safeguards are typically used. For GMI cuts, the most important ones are a lower bound on *integer infeasibility* and an upper bound on *dynamism*. For a solution $\bar{x}$, define the *integer infeasibility* of an integer variable x_i as $\min\{\bar{x}_i - \lfloor \bar{x}_i \rfloor, \lceil \bar{x}_i \rceil - \bar{x}_i\}$. We do not generate a cut from a fractional row whose basic integer variable has an integer infeasibility smaller than 0.01. Define the *dynamism* of a cut as the ratio between the largest and the smallest nonzero absolute values of its left hand side coefficients. We discard generated cuts whose dynamism is larger than 10^9. We use the same parameters for the generation of two-constraint cuts with, in addition, the condition that, for a row to be considered an integer row, the integer infeasibility of its basic integer variable must be at most 10^{-5}.

A method for testing the accuracy and strength of cut generators is given in [53]. Next we give a short description of the method. Please refer to the original paper for additional details.

Consider an instance I with a known feasible solution x^*. Define a *cutting step* as the operation of adding cuts obtained from a cut generator and checking if x^* is

still feasible (report failure of the cut generator if x^* is no longer feasible). Define a *branching step* as getting an optimal solution $\bar{x}$ of the current LP relaxation, picking at random an integer variable x_i with a fractional value $\bar{x}_i$ and imposing in the LP that its value must be x_i^*.

The method, called a *dive towards a feasible solution* x^*, amounts to repeatedly performing a cutting step followed by a branching step until an integer feasible solution of the LP relaxation is obtained. In our experiments x^* is chosen to be a known feasible solution whose objective value is likely to be optimum or close to optimum. A number of dives (we use twenty dives in our experiments) are performed for each instance I. In our experiments, we set a time limit of 3 hours for each dive.

The motivation for using cuts in a branch-and-cut algorithm is to improve the bound obtained from the linear relaxation and to reduce the size of the enumeration tree. With these two goals in mind, we track the following during each dive:

- Fraction of the integrality gap closed after some given numbers of branching steps;
- Number of branching steps required to close some given fractions of the integrality gap.

The random branching step allows for meaningful statistical analysis of these two performance measures. We used the nonparametric *Quade* test as described in [29]. All tests are done with a confidence level of 95%. The statistical package `R` [3] version 2.10.0 (2009-10-26) is used for the statistical analysis of the results.

In Table 6.1, we give the average gap closed by the four algorithms `G`, `G-2Rounds`, `G+Allpairs`, `G+Deepest` after 0, 4, 8 and 12 branching steps in a dive. The following problems timed out without any results for at least one generator: 10teams_c, air04_c, air05_c, arki_c, dano3mip_c, fast0507_c, mitre_c, mkc_c, seymour_c. The table gives results for the remaining 59 instances.

We make the following observations from our results.

- Comparing `G` and `G+Allpairs`, we note that triangle cuts only improve marginally over the GMI cuts at depths 0, 4 and 8. The improvement in average gap closed at these three depths is less than 1%. At depth 12, the average gap closed is actually more for `G`, but this difference is not statistically significant according to the Quade test.

Table 6.1: Average percentage of gap closed at different depths

	Depth 0 (root node)	Depth 4	Depth 8	Depth 12
`G`	28.33	57.41	68.15	75.23
`G+Allpairs`	29.11	58.13	68.43	74.81
`G+Deepest`	28.80	58.09	68.59	75.40
`G-2Rounds`	36.66	59.41	68.75	75.47

Even restricting to instances that are degenerate at the root, the improvement is only 1.2% on average at the root which is disappointingly small given that we are generating at least four times as many cuts.

- Comparing `G+Allpairs` and `G+Deepest`, we note that most of the above improvement at depths 0, 4 and 8 can be achieved using only the deepest triangles. In fact, surprisingly, at depths 8 and 12 `G+Deepest` closes more average gap than `G+Allpairs`, and at depth 12 the difference is statistically significant. This may be due to the fact that adding too many cuts deteriorates the numerical properties of the basis, and this has an adverse effect on the quality of the cuts at greater depths.

- Comparing `G-2Rounds` and `G+Deepest`, which generate roughly the same number of cuts, we conclude that it is clearly better to use GMI cuts at levels 0 and 4. This is confirmed by the Quade test which shows that the difference is statistically significant. At depths 8 and 12 the difference is not statistically significant.

The results from Table 6.1 seem to indicate that our family of two-constraint cuts is not competitive with GMI cuts in terms of gap closed. This conclusion is confirmed by statistical tests performed on the second performance measure introduced above. For brevity, the detailed results of these tests are not reported here.

We performed other experiments which seem to suggest that among the two-constraint cuts that we generate, the split cuts are the more useful ones. First, we considered two variants of `G+Deepest` where the choice of the cut is based on the length of the segment p^2p^3 as defined in Section 5.6.3. In the first variant, we chose the cut with the longest segment p^2p^3; whereas, in the second we chose the shortest. The motivation for the first choice comes from the theoretical results in Chapter 5, where it is shown that the gap closed by the split closure is smaller when the segment p^2p^3 is long. However, surprisingly, the second variant closes

more gap on average at levels 0, 4, 8 and 12, although the differences are small. This may be explained from the fact that the second variant generates more splits than the first and, in fact, at the root it generates almost only splits.

In another experiment, we tried to see whether our two-constraint cuts could cut off the optimum vertex of the split closure, as computed by Balas and Saxena [16]. For this purpose we used the formulations in `OSCLIB 1.0` [4]. In all but 2 of these instances, we could not cut off the split optimum.

We also compared the depths of the GMI cuts with our two-constraint cuts. For 15 instances, no triangle cuts are generated at the root because the optimal tableau is not degenerate and there are no basic integer variables with integer values. For the instance nw04_c, the average depths are too small for any meaningful comparison. For each of the 52 remaining instances, we measure the depth of the deepest triangle cut as a percentage of the depth of the GMI cut from the same fractional row. We then average this percentage over all fractional rows to get a single percentage number associated with that instance. For 46 out of the 52 instances, this percentage is between 99 and 125. The other 6 have a percentage between 125 and 200. This shows that, in general, the deepest triangle cuts are deeper than the corresponding GMI cuts.

6.3.2 Importance of the experimental setup

In this section, we show that one can reach drastically different conclusions by modifying the experimental setup in a seemingly natural way. Instead of using our own GMI cut generator, we now use the GMI cut generator `CglGomory` from the COIN-OR library with default settings. The table corresponding to Table 6.1 is as follows :

Table 6.2: Average percentage of gap closed at different depths

	Depth 0 (root node)	Depth 4	Depth 8	Depth 12
`CglGomory`	23.83	51.86	62.91	69.83
`CglGomory+All pairs`	24.72	53.18	64.37	71.68
`CglGomory+Deepest`	24.45	53.92	64.54	71.19
`CglGomory-2Rounds`	30.89	53.73	62.85	69.97

The conclusions we reach from this table are different from those reached from Table 6.1. We conclude that `CglGomory+All pairs` is significantly better

than CglGomory at all depths and this advantage increases with depth, with the difference being almost 2% at depth 12. These differences are statistically significant according to the Quade test. Moreover, CglGomory+Deepest is superior to CglGomory-2Rounds at depths 4, 8 and 12 with respect to gap closed and this difference is statistically significant at depths 8 and 12. When comparison is made on the number of branching steps needed to close different fractions of the gap, CglGomory+Deepest is significantly better than CglGomory-2Rounds for closing 70% or more of the gap according to the Quade test.

How do we reconcile the different conclusions reached from Tables 6.1 and 6.2 ?

Although our goal in this paper is to compare the strength of different cut families, in practice we can only compare cut generators. The generator G described in the previous section and CglGomory are two different generators for GMI cuts. We described above the parameter settings for accepting the cuts in G. For the default CglGomory generator the bound on integer infeasibility is 0.05, the bound on dynamism is 10^9, and there are several other parameters that may affect the acceptance or rejection of a GMI cut. Even if we set the integer infeasibility and dynamism bounds to the same value in both generators, there are still significant differences in these two cut generators. This raises the question of which generator to use for our experiments and how to set the parameters. Depending on the parameter settings, a cut generator may reject cuts more often (conservative strategy) or less frequently (aggressive strategy).

A typical way that researchers test new cut generators is to add their generator to existing base generators and evaluate the difference in performance. In such a setting, if the base generator is conservative and the tested generator is aggressive, one may expect good results. On the other hand, if the base generator is made more aggressive or the tested generator is made more conservative, these improvements might be eroded to a large extent. The difference between Tables 6.1 and 6.2 is a concrete example of this phenomenon. This issue is usually not addressed in the literature. There is a trade-off between reliability of cut generators and their aggressiveness. Therefore, a key aspect of such experimental setups should involve measuring the reliability of cut generators. This was addressed by Margot in [53]. The general philosophy is to put the generator under stress and to record the number of failures due to invalid cut generation. Concretely, the reliability of a cut generator is tested using the diving experiment described in the previous section where 10 rounds of cuts are generated before a branching step is performed. According to this test, G is comparable in reliability to CglGomory. In fact, G never

generated an invalid cut for the 59 instances of MIPLIB3_C_V2 considered in this paper; whereas, CglGomory generated invalid cuts on the instance harp2_c.

In our view, G should be used over CglGomory in our experiments because it is more aggressive and equally reliable. Another advantage of G over CglGomory is that not only are the parameter settings the same as those for the two-constraint cut generators, the computer code itself is the same. This does not necessarily imply that G should be preferred to CglGomory in a general branch-and-cut setting. More experiments would be needed to assess the reliability of the two cut generators.

This discussion shows that the setup based on CglGomory is flawed and that the correct setup is the one used in Section 6.3.1, if the goal is to evaluate our two-constraint cuts as compared to GMI cuts. The conclusion is that our family of two-constraint cuts is not competitive with GMI cuts.

Chapter 7

A Counterexample to a Conjecture of Gomory and Johnson

In this chapter, we will present results for the valid inequalities for the infinite group problem (2.11). As noted in Section 2.5.1, this problem was studied by Gomory and Johnson in a series of papers [43, 44, 48, 46], most notably the case where $k = 1$ in the system (2.11). We recall the model below.

Let $f \in]0, 1[$ be given. Consider the following infinite group problem with a single equality constraint and a nonnegative integer variable s_r associated with each real number $r \in [0, 1[$.

$$\begin{array}{rcl} \sum_{r\in[0,1[} r s_r & = & f \\ s_r & \in & \mathbb{Z}_+ \qquad \forall r \in [0,1[\\ & & s \text{ has finite support,} \end{array} \tag{7.1}$$

where additions are performed modulo 1. Vector s has *finite support* if $s_r \neq 0$ for a finite number of distinct $r \in [0, 1[$.

Gomory and Johnson [44] say that a function $\pi : [0, 1[\to \mathbb{R}$ is a *valid function* for (7.1) if π is nonnegative, $\pi(0) = 0$, and every solution of (7.1) satisfies

$$\sum_{r\in[0,1[} \pi(r) s_r \geq 1.$$

Note that, if π is a valid function, then $\pi(f) \geq 1$. This is analogous to the concept of a valid function defined in the context of studying the models (3.1) and (3.9) in Chapter 3.

For any valid function π, let $P(\pi)$ be the set of solutions which satisfy $\sum_{r\in[0,1[}\pi(r)s_r \geq 1$ at equality. An inequality π is a *facet* for (7.1) if and only if $P(\pi^*) \supset P(\pi)$ implies $\pi^* = \pi$ for every valid function π^*.

A function is *piecewise linear* if there are finitely many values $0 = r_0 < r_1 < \ldots < r_k = 1$ such that the function is of the form $\pi(r) = a_j r + b_j$ in interval $[r_{j-1}, r_j[$, for $j = 1, \ldots k$. The *slopes* of a piecewise linear function are the different values of a_j for $j = 1, \ldots k$.

Gomory and Johnson [46] gave many examples of facets, and in all their examples the facets are piecewise linear. This led them to formulate the following conjecture, which they describe as "important and challenging".

Conjecture 99 (Facet Conjecture)**.** *Every continuous facet for* (7.1) *is piecewise linear.*

We show that the above conjecture is false by exhibiting a facet for (7.1) that is continuous but not piecewise linear. This was done in collaboration with Michele Conforti, Gerard Cornuejols and Giacomo Zambelli [20].

Literature overview

The definition of valid function we stated, which is the one given by Gomory and Johnson in [44], differs from the one adopted later by the same two authors in [46], where they included in the definition the further assumption that the function π be continuous. So a "valid function" in [46] corresponds to a "continuous valid function" from [44] and the present paper. Dey et al. [36] show that there are facets that are not continuous.

The definition of facet given in our paper is identical to the one given by Gomory and Johnson in [46], with the caveat that in [46] valid functions are required to be continuous. Thus, given a continuous valid function π, π would be a facet according to the definition in [46] if and only if $P(\pi^*) \supset P(\pi)$ implies $\pi^* = \pi$ for every *continuous* valid function π^*. For clarity, we refer to a continuous valid function satisfying the latter property as a *facet in the sense of Gomory-Johnson*. By definition, if a continuous valid function is a facet, it is also a facet in the sense of Gomory-Johnson. Thus, the conjecture they state in [46] is actually that *every facet in the sense of Gomory-Johnson is piecewise linear*. The above discussion shows that a counterexample to Conjecture 99 also disproves the conjecture in [46].

In earlier work, Gomory and Johnson [44] emphasized extreme functions rather than facets. A valid function π is *extreme* if it cannot be expressed as a convex combination of two distinct valid functions. It follows from the definition that facets are extreme. Therefore our counterexample also provides an extreme function that is continuous but not piecewise linear. Gomory and Johnson [46] write in a footnote that the definition of facet "is different from, although eventually equivalent to," the definition of extreme function. The statement that extreme functions are facets appears to be quite nontrivial to prove, and to the best of our knowledge there is no proof in the literature. We therefore cautiously treat extreme functions and facets as distinct concepts, and leave their equivalence as an open question.

We obtain the counterexample to Conjecture 99 by exhibiting a sequence of piecewise linear functions and then considering the pointwise limit of this sequence of functions. These functions were discovered by Kianfar and Fathi [51] who show that they are facets for the infinite group problem. They call them the n-step MIR (Mixed-Integer Rounding) functions in their paper. Our treatment and analysis in this paper is different from theirs. The emphasis in [51] was on deriving valid inequalities for MILPs which are generalizations of the standard MIR inequalities. In this paper, we use this class of functions primarily to construct a counterexample to Conjecture 99.

7.1 Preliminaries

A valid function $\pi : [0,1[\to \mathbb{R}$ is *minimal* if there is no valid function π' such that 1) $\pi'(a) \leq \pi(a)$ for all $a \in [0,1[$, and 2) the inequality is strict for at least one a. If π is a minimal valid function, then $\pi(r) \leq 1$ for every $r \in [0,1[$, as follows immediately from the fact that π is nonnegative.

When convenient, we will extend the domain of definition of the function π to the whole real line $\mathbb{R}$ by making the function periodic: $\pi(x) = \pi(x+k)$ for any $x \in [0,1[$ and $k \in \mathbb{Z}$.

A function $\pi : \mathbb{R} \to \mathbb{R}$ is *subadditive* if for every $a, b \in \mathbb{R}$

$$\pi(a+b) \leq \pi(a) + \pi(b).$$

Given $f \in]0,1[$, a function $\pi : [0,1[\to \mathbb{R}$ is *symmetric* if for every $a \in [0,1[$

$$\pi(a) + \pi(f-a) = 1.$$

Gomory and Johnson prove the following result in [44].

Theorem 100 (Minimality Theorem)**.** *Let $\pi : [0, 1[\to \mathbb{R}$ be such that $\pi(0) = 0$ and $\pi(f) = 1$. A necessary and sufficient condition for π to be valid and minimal is that π is subadditive and symmetric.*

Any facet for (7.1) is minimal. Therefore if π is a facet for (7.1), then π is subadditive and symmetric. The following two facts are well-known and will be useful in our arguments.

Fact 101. *If π_1 and π_2 are subadditive, then $\pi_1 + \pi_2$ is subadditive.*

Fact 102. *Let π be a subadditive function and define $\pi'(x) = \alpha\pi(\beta x)$ for some constants $\alpha > 0$ and β. Then π' is subadditive.*

Proof. $\pi'(a + b) = \alpha\pi(\beta(a + b)) \leq \alpha(\pi(\beta a) + \pi(\beta b)) = \pi'(a) + \pi'(b)$. □

A function π' defined as $\pi'(x) = \alpha\pi(\beta x)$ will be referred to as a *scaling* of π.

Let $E(\pi)$ denote the set of all possible of inequalities $\pi(u_1) + \pi(u_2) \geq \pi(u_1 + u_2)$ that are satisfied as *equalities* by π. Here u_1 and u_2 are any real numbers. The following theorem is proved in Gomory and Johnson [46] and is used in this paper to prove that certain inequalities are facets.

Theorem 103 (Facet Theorem)**.** *Let π be a minimal valid function. If there is no minimal valid function that satisfies the equalities in $E(\pi)$ other than π itself, then π is a facet.*

We remark that, even though Theorem 103 is proved in [46] under the assumptions that valid functions are continuous, the continuity assumption is not needed in the proof, thus the statement remains true even in the setting of the present paper.

In the paper we need the following lemma, which is a variant of the Interval Lemma stated in Gomory and Johnson [46]. They prove the lemma under the assumption that the function in the statement is continuous, whereas we only require the function to be bounded one every interval. Other variants of the Interval Lemma that do not require the function to be continuous have been given by Dey et al. [36]. The proof we give is in the same spirit of the solution of Cauchy's Equation (see for example Chapter 2 of Aczél [5]). The author would like to thank Santanu Dey for fruitful discussions in this regard.

Lemma 104 (Interval Lemma)**.** *Let $\pi : \mathbb{R} \to \mathbb{R}$ be a function bounded on every bounded interval. Given real numbers $u_1 < u_2$ and $v_1 < v_2$, let $U = [u_1, u_2]$, $V = [v_1, v_2]$, and $U + V = [u_1 + v_1, u_2 + v_2]$.*
If $\pi(u) + \pi(v) = \pi(u + v)$ for every $u \in U$ and $v \in V$, then there exists $c \in \mathbb{R}$ such that $\pi(u) = \pi(u_1) + c(u - u_1)$ for every $u \in U$, $\pi(v) = \pi(v_1) + c(v - v_1)$ for every $v \in V$, $\pi(w) = \pi(u_1 + v_1) + c(w - u_1 - v_1)$ for every $w \in U + V$.

Proof. We first show the following.

Claim 1. Let $u \in U$, and let $\varepsilon > 0$ such that $v_1 + \varepsilon \in V$. For every nonnegative integer p such that $u + p\varepsilon \in U$, we have $\pi(u + p\varepsilon) - \pi(u) = p(\pi(v_1 + \varepsilon) - \pi(v_1))$.

For $h = 1 \ldots, p$, by hypothesis $\pi(u + h\varepsilon) + \pi(v_1) = \pi(u + h\varepsilon + v_1) = \pi(u + (h - 1)\varepsilon) + \pi(v_1 + \varepsilon)$. Thus $\pi(u + h\varepsilon) - \pi(u + (h - 1)\varepsilon) = \pi(v_1 + \varepsilon) - \pi(v_1)$, for $h = 1, \ldots, p$. By summing the above p equations, we obtain $\pi(u + p\varepsilon) - \pi(u) = p(\pi(v_1 + \varepsilon) - \pi(v_1))$. This concludes the proof of Claim 1.

Let $\bar{u}, \bar{u}' \in U$ such that $\bar{u} - \bar{u}' \in \mathbb{Q}$ and $\bar{u} > \bar{u}'$. Define $c := \frac{\pi(\bar{u}) - \pi(\bar{u}')}{\bar{u} - \bar{u}'}$.

Claim 2. For every $u, u' \in U$ such that $u - u' \in \mathbb{Q}$, we have $\pi(u) - \pi(u') = c(u - u')$.

We only need to show that, given $u, u' \in U$ such that $u - u' \in \mathbb{Q}$, we have $\pi(u) - \pi(u') = c(u - u')$. We may assume $u > u'$. Choose a positive rational ε such that $\bar{u} - \bar{u}' = \bar{p}\varepsilon$ for some integer $\bar{p}$, $u - u' = p\varepsilon$ for some integer p, and $v_1 + \varepsilon \in V$. By Claim 1,

$$\pi(\bar{u}) - \pi(\bar{u}') = \bar{p}(\pi(v_1 + \varepsilon) - \pi(v_1)) \quad \text{and} \quad \pi(u) - \pi(u') = p(\pi(v_1 + \varepsilon) - \pi(v_1)).$$

Dividing the last equality by $u - u'$ and the second to last by $\bar{u} - \bar{u}'$, we get

$$\frac{\pi(v_1 + \varepsilon) - \pi(v_1)}{\varepsilon} = \frac{\pi(\bar{u}) - \pi(\bar{u}')}{\bar{u} - \bar{u}'} = \frac{\pi(u) - \pi(u')}{u - u'} = c.$$

Thus $\pi(u) - \pi(u') = c(u - u')$. This concludes the proof of Claim 2.

Claim 3. For every $u \in U$, $\pi(u) = \pi(u_1) + c(u - u_1)$.

Let $\delta(x) = \pi(x) - cx$. We show that $\delta(u) = \delta(u_1)$ for all $u \in U$ and this proves the claim. Since π is bounded on every bounded interval, δ is bounded over U, V and $U + V$. Let M be a number such that $|\delta(x)| \leq M$ for all $x \in U \cup V \cup (U + V)$.

Suppose by contradiction that, for some $u^* \in U$, $\delta(u^*) \neq \delta(u_1)$. Let N be a positive integer such that $|N(\delta(u^*) - \delta(u_1))| > 2M$.
By Claim 2, $\delta(u^*) = \delta(u)$ for every $u \in U$ such that $u^* - u$ is rational. Thus there exists $\bar{u}$ such that $\delta(\bar{u}) = \delta(u^*)$, $u_1 + N(\bar{u} - u_1) \in U$ and $v_1 + \bar{u} - u_1 \in V$. Let $\bar{u} - u_1 = \varepsilon$. By Claim 1,

$$\delta(u_1 + N\varepsilon) - \delta(u_1) = N(\delta(v_1 + \varepsilon) - \delta(v_1) = N(\delta(u_1 + \varepsilon) - \delta(u_1)) = N(\delta(\bar{u}) - \delta(u_1))$$

Thus $|\delta(u_1 + N\varepsilon) - \delta(u_1)| = |N(\delta(\bar{u}) - \delta(u_1))| = |N(\delta(u^*) - \delta(u_1))| > 2M$, which implies $|\delta(u_1 + N\varepsilon)| + |\delta(u_1)| > 2M$, a contradiction. This concludes the proof of Claim 3.

By symmetry between U and V, Claim 3 implies that there exists some constant c' such that, for every $v \in V$, $\pi(v) = \pi(v_1) + c'(v - v_1)$. We show $c' = c$. Indeed, given $\varepsilon > 0$ such that $u_1 + \varepsilon \in U$ and $v_1 + \varepsilon \in V$, $c\varepsilon = \pi(u_1 + \varepsilon) - \pi(u_1) = \pi(v_1 + \varepsilon) - \pi(v_1) = c'\varepsilon$, where the second equality follows from Claim 1.
Therefore, for every $v \in V$, $\pi(v) = \pi(v_1) + c\pi(v - v_1)$. Finally, since $\pi(u) + \pi(v) = \pi(u + v)$ for every $u \in U$ and $v \in V$, for every $w \in U + V$, $\pi(w) = \pi(u_1 + v_1) + c(w - u_1 - v_1)$. □

The following theorem, due to Gomory and Johnson [46], gives a class of facets.

Theorem 105. *Let $\pi : [0, 1[\to \mathbb{R}$ be a minimal valid function that is piecewise linear. If π has only two distinct slopes, then π is a facet.*

Again, we note that Theorem 105 is proved in [46] under the assumptions that valid functions are continuous. However, in [46] the continuity assumption is used in the proof only when applying the Interval Lemma. Since our version of the Interval Lemma (Lemma 104) applies to any bounded function, and since minimal valid functions are bounded, Theorem 105 is valid also in the setting of the present paper. Gomory and Johnson give in [45] a very similar statement to the one of Theorem 105, namely that piecewise linear minimal valid functions with only two distinct slopes are extreme.

7.2 The construction

We first define a sequence of valid functions $\psi_i : [0, 1[\to \mathbb{R}$ that are piecewise linear, and then consider the limit ψ of this sequence. We will then show that ψ is a facet but not piecewise linear.

Let $0 < \alpha < 1$. ψ_0 is the triangular function given by

$$\psi_0(x) = \begin{cases} \frac{1}{\alpha}x & 0 \leq x \leq \alpha \\ \frac{1-x}{1-\alpha} & \alpha \leq x < 1. \end{cases}$$

Notice that the corresponding inequality $\sum_{r\in[0,1[} \psi_0(r)s_r \geq 1$ defines the Gomory mixed-integer inequality if one views (7.1) as a relaxation of the simplex tableau of an integer program.

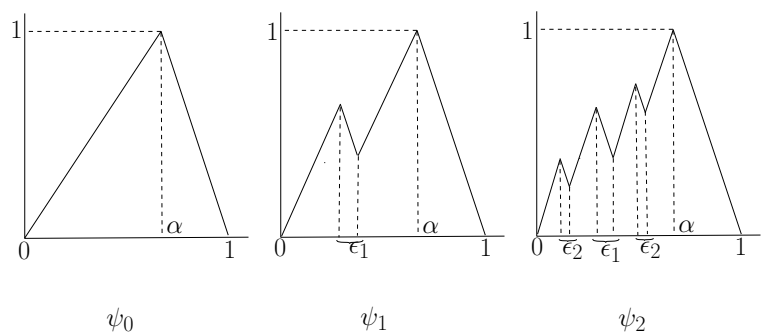


Figure 7.1: First two steps in the construction of the limit function

We first fix a nonincreasing sequence of positive real numbers ϵ_i, for $i = 1, 2, 3, \ldots$, such that $\epsilon_1 \leq 1 - \alpha$ and

$$\sum_{i=1}^{+\infty} 2^{i-1}\epsilon_i < \alpha. \tag{7.2}$$

For example, $\epsilon_i = \alpha(\frac{1}{4})^i$ is such a sequence when $0 < \alpha \leq \frac{4}{5}$. The upper bound of $\frac{4}{5}$ for α is implied by the fact that $\epsilon_1 \leq 1 - \alpha$.

We construct ψ_{i+1} from ψ_i by modifying each segment with positive slope in the graph of ψ_i as follows.

For every maximal (with respect to set inclusion) interval $[a, b] \subseteq [0, \alpha]$ where ψ_i has constant positive slope we replace the line segment from $(a, \psi_i(a))$ to $(b, \psi_i(b))$ with the following three segments.

- The segment connecting $(a, \psi_i(a))$ and $(\frac{(a+b)-\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) + \frac{\epsilon_{i+1}}{2(1-\alpha)})$,

- The segment connecting $(\frac{(a+b)-\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) + \frac{\epsilon_{i+1}}{2(1-\alpha)})$ and $(\frac{(a+b)+\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) - \frac{\epsilon_{i+1}}{2(1-\alpha)})$,
- The segment connecting $(\frac{(a+b)+\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) - \frac{\epsilon_{i+1}}{2(1-\alpha)})$ and $(b, \psi_i(b))$.

Figure 7.1 shows the transformation of ψ_0 to ψ_1 and ψ_1 to ψ_2.

The function ψ which we show to be a facet but not piecewise linear is defined as the limit of this sequence of functions, namely

$$\psi(x) = \lim_{i\to\infty} \psi_i(x) \tag{7.3}$$

This limit is well defined when (7.2) holds, as shown in Section 7.4.

In the next section we show that each function ψ_i is well defined and is a facet. In Section 7.4 we analyze the limit function ψ, showing that it is well defined, is a facet, but is not piecewise linear.

As discussed in the Introduction, the sequence ψ_i defines a class of facets which was also discovered independently by Kianfar and Fathi in [51] where they are referred to as n-step MIR functions. Their constructions are a little more general than the class of facets defined by the ψ_i's. Our analysis in the next section is different from their treatment of these functions.

7.3 Analysis of the function ψ_i

Fact 106. *For $i \geq 0$, ψ_i is a continuous function which is piecewise linear with 2^i pieces with positive slope and 2^i pieces with negative slope. Furthermore:*

1. *There is one negative slope interval of length $1-\alpha$ and there are 2^{k-1} negative slope intervals of length ϵ_k for $k = 1, \ldots, i$;*
2. *The negative slope pieces have slope $-\frac{1}{1-\alpha}$;*
3. *Each positive slope interval has length $\frac{\gamma_i}{2^i}$, where $\gamma_i = \alpha - \sum_{k=1}^{i} 2^{k-1}\epsilon_k$;*
4. *The positive slope pieces have slope $\frac{1-\gamma_i}{(1-\alpha)\gamma_i}$;*
5. *The function ψ_i is well-defined.*

Proof. The fact that ψ_i is a continuous function which is piecewise linear with 2^i pieces with positive slope and 2^i pieces with negative slope, and facts 1. and 2. are immediate by construction.
Therefore the sum of the lengths of the negative slope intervals is $1-\alpha+\sum_{k=1}^{i} 2^{k-1}\epsilon_k$. Since ψ_i contains 2^i positive slope intervals with the same length, this proves 3.
The total decrease of ψ_i in $[0,1]$ is $\frac{-1}{1-\alpha}(1-\gamma_i)$. Since ψ_i is continuous, piecewise linear, all positive slope intervals have the same slope and $\psi_i(0)=\psi_i(1)=0$, then a positive slope interval has slope $\frac{1-\gamma_i}{(1-\alpha)\gamma_i}$ and this proves 4.
Finally, by (7.2), $\gamma_i > 0$ for every $i \geq 0$, thus ψ_i is a well-defined function.

□

We now demonstrate that each function ψ_i is subadditive. Note that the function ψ_i depends only upon the choice of parameters $\alpha, \epsilon_1, \epsilon_2, \ldots, \epsilon_i$. It will sometimes be convenient to denote the function ψ_i by $\psi_i^{\alpha,\epsilon_1,\epsilon_2,\ldots,\epsilon_i}$ in this section.

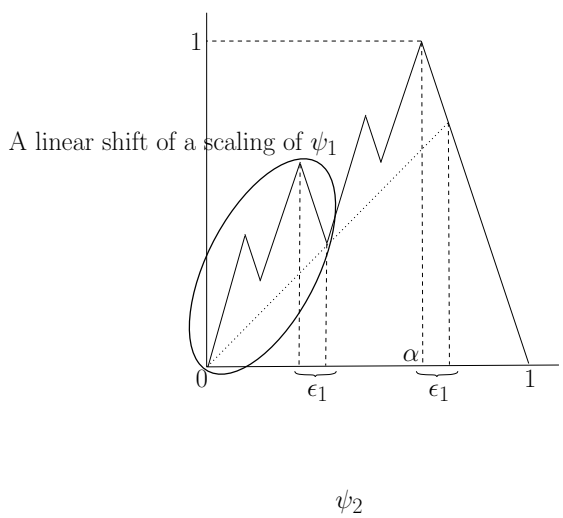


Figure 7.2: Illustrating the proof of subadditivity of ψ_2

The key observation is the following lemma. Figure 7.2 illustrates this for the function $\psi_2^{\alpha,\epsilon_1,\epsilon_2}$.

Lemma 107. *For $x \in [0, \alpha+\epsilon_1]$ and $i \geq 1$, $\psi_i^{\alpha,\epsilon_1,\epsilon_2,\ldots,\epsilon_i}(x) = \lambda x + \mu \psi_{i-1}^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1},\frac{2\epsilon_2}{\alpha+\epsilon_1},\frac{2\epsilon_3}{\alpha+\epsilon_1},\ldots,\frac{2\epsilon_i}{\alpha+\epsilon_1}}(\frac{2x}{\alpha+\epsilon_1})$, where*

$$\lambda = \frac{1-\alpha-\epsilon_1}{(\alpha+\epsilon_1)(1-\alpha)} \textit{and } \mu = \frac{\epsilon_1}{(\alpha+\epsilon_1)(1-\alpha)}.$$

Proof. Notice that λ is the slope of the line passing through the points $(0,0)$ and $(\frac{\alpha+\epsilon_1}{2}, \psi_1^{\alpha,\epsilon_1}(\frac{\alpha+\epsilon_1}{2})) = (\frac{\alpha+\epsilon_1}{2}, \frac{1-\epsilon_1}{2})$.

For $x \in [0, \alpha+\epsilon_1]$, let $\phi_i(x) = \psi_i^{\alpha,\epsilon_1,\epsilon_2,\ldots,\epsilon_i}(x) - \lambda x$. Notice that the graph of ϕ_1 in the interval $[0, \alpha+\epsilon_1]$ is comprised of two identical triangles, one with basis $[0, \frac{\alpha+\epsilon_1}{2}]$ and apex $(\frac{\alpha-\epsilon_1}{2}, \mu)$ and the other with basis $[\frac{\alpha+\epsilon_1}{2}, \alpha+\epsilon_1]$ and apex (α, μ), where $\mu = \psi_1(\frac{\alpha-\epsilon_1}{2}) - \lambda(\frac{\alpha-\epsilon_1}{2})$. Therefore, for $x \in [0,1[$,

$$\mu^{-1}\phi_1\left(\frac{(\alpha+\epsilon_1)x}{2}\right) = \psi_0^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1}}(x)$$

thus $\phi_1(x) = \mu\psi_0^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1}}(\frac{2x}{\alpha+\epsilon_1})$ because $\phi_1(x) = \phi_1(x + \frac{\alpha+\epsilon_1}{2})$ for every $x \in [0, \frac{\alpha+\epsilon_1}{2}[$.

Assume by induction that $\phi_i(x) = \phi_i(x + \frac{\alpha+\epsilon_1}{2})$ for every $x \in [0, \frac{\alpha+\epsilon_1}{2}[$, and that, for $x \in [0,1[$, $\mu^{-1}\phi_i\left(\frac{(\alpha+\epsilon_1)x}{2}\right) = \psi_{i-1}^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1}, \frac{2\epsilon_2}{\alpha+\epsilon_1}, \ldots, \frac{2\epsilon_i}{\alpha+\epsilon_1}}(x)$.

Notice that, by Fact 106, the slope of ψ_i in the intervals of positive slope is always greater than λ, hence ϕ_i has positive slope exactly in the same intervals where ψ_i has positive slope.

Therefore, by construction of $\psi_{i+1}^{\alpha,\epsilon_1,\ldots,\epsilon_{i+1}}$, the function ϕ_{i+1} is obtained from ϕ_i by replacing each maximal positive slope segment $[(a, \phi_i(a)), (b, \phi_i(b))]$ with:
- the segment connecting $(a, \phi_i(a))$ and $(\frac{(a+b)-\epsilon_{i+1}}{2}, \phi_i(\frac{a+b}{2}) + \frac{\epsilon_{i+1}}{2(1-\alpha)})$,
- the segment connecting $(\frac{(a+b)-\epsilon_{i+1}}{2}, \phi_i(\frac{a+b}{2}) + \frac{\epsilon_{i+1}}{2(1-\alpha)})$ and $(\frac{(a+b)+\epsilon_{i+1}}{2}, \phi_i(\frac{a+b}{2}) - \frac{\epsilon_{i+1}}{2(1-\alpha)})$,
- the segment connecting $(\frac{(a+b)+\epsilon_{i+1}}{2}, \phi_i(\frac{a+b}{2}) - \frac{\epsilon_{i+1}}{2(1-\alpha)})$ and $(b, \phi_i(b))$.

Thus, by induction, $\phi_{i+1}(x) = \phi_{i+1}(x + \frac{\alpha+\epsilon_1}{2})$ for every $x \in [0, \frac{\alpha+\epsilon_1}{2}[$ and, for $x \in [0,1[$, we have that $\mu^{-1}\phi_{i+1}\left(\frac{(\alpha+\epsilon_1)x}{2}\right) = \psi_i^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1}, \frac{2\epsilon_2}{\alpha+\epsilon_1}, \ldots, \frac{2\epsilon_i}{\alpha+\epsilon_1}, \frac{2\epsilon_{i+1}}{\alpha+\epsilon_1}}(x)$. Therefore, for $x \in [0, \alpha+\epsilon_1[$, we have $\phi_{i+1}(x) = \mu\psi_i^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1}, \frac{2\epsilon_2}{\alpha+\epsilon_1}, \ldots, \frac{2\epsilon_{i+1}}{\alpha+\epsilon_1}}(\frac{2x}{\alpha+\epsilon_1})$. □

Remark 108. *Given any $0 < \alpha < 1$ and any non-increasing sequence of positive real numbers ϵ_i satisfying (7.2) and $\epsilon_1 \leq 1-\alpha$, let $\alpha' = \frac{\alpha-\epsilon_1}{\alpha+\epsilon_1}$, $\epsilon_i' = \frac{2\epsilon_{i+1}}{\alpha+\epsilon_1}$, $i \geq 1$. Then $\epsilon_1' \leq 1-\alpha'$, $\{\epsilon_i'\}$ is a non-increasing sequence, and $\sum_{i=1}^{+\infty} 2^{i-1}\epsilon_i' < \alpha'$.*

We next prove that each ψ_i is a non-negative function.

Fact 109. *$\psi_i^{\alpha,\epsilon_1,\ldots\epsilon_i}(x) \geq 0$ for all x, and for all parameters such that $\epsilon_1 \leq 1-\alpha$ and ϵ_i is a non-increasing sequence.*

Proof. The proof is by induction on i. ψ_0 is non-negative by definition.

Consider ψ_{i+1}. Clearly $\psi_{i+1}(x) \geq 0$ for $x \in [\alpha + \epsilon_1, 1[$, since $\psi_{i+1}(x) = \psi_0(x)$ in this interval. Note that in Lemma 107, $\lambda \geq 0$ because $1 - \alpha \geq \epsilon_1$. So, when $x \in [0, \alpha + \epsilon_1]$ Lemma 107 implies that ψ_{i+1} is non-negative, because ψ_i is non-negative. Note that the parameters for ψ_i also satisfy the hypothesis by Remark 108, so we can use the induction hypothesis. □

Lemma 110. *Given any $0 < \alpha < 1$ and any nonincreasing sequence of positive real numbers ϵ_i satisfying (7.2) and $\epsilon_1 \leq 1 - \alpha$, the function $\psi_i^{\alpha,\epsilon_1,\epsilon_2,\ldots,\epsilon_i}$ is subadditive for all i.*

Proof. The proof is by induction. ψ_0^α is subadditive, since it is a valid and minimal Gomory function. By the induction hypothesis, $\psi_k^{\alpha,\epsilon_1,\ldots,\epsilon_k}$ is subadditive and we wish to show this implies that $\psi_{k+1}^{\alpha,\epsilon_1,\ldots,\epsilon_{k+1}}$ is subadditive.

By Remark 108 and induction, the function $\psi_k^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1},\frac{2\epsilon_2}{\alpha+\epsilon_1},\frac{2\epsilon_3}{\alpha+\epsilon_1},\ldots,\frac{2\epsilon_{k+1}}{\alpha+\epsilon_1}}$ is subadditive.

We define the function $\psi'_k(x) = \mu\psi_k^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1},\frac{2\epsilon_2}{\alpha+\epsilon_1},\frac{2\epsilon_3}{\alpha+\epsilon_1},\ldots,\frac{2\epsilon_{k+1}}{\alpha+\epsilon_1}}(\frac{2x}{\alpha+\epsilon_1})$ where μ is defined in the statement of Lemma 107. Note that ψ'_k has a period of $\frac{\alpha+\epsilon_1}{2}$ and ψ'_k is subadditive by Fact 102.

In the remaining part of the proof, we will not need the extended notation $\psi_k^{\alpha,\epsilon_1,\epsilon_2,\ldots,\epsilon_k}$ and we shall refer to this function as simply ψ_k. We now prove the subadditivity of ψ_{k+1} assuming the subadditivity of ψ_k, i.e. $\psi_{k+1}(a+b) \leq \psi_{k+1}(a) + \psi_{k+1}(b)$. Assume without loss of generality that $a \leq b$. We then have the following two cases.

Case 1 : b is in the range $[0, \alpha + \epsilon_1]$.
If $a+b \in [0, \alpha+\epsilon_1]$, $\psi_{k+1}(a+b) = \psi'_k(a+b) + \lambda(a+b)$ by Lemma 107. Now Fact 101 shows that ψ_{k+1} is subadditive.
If $a+b$ is in the range $[\alpha+\epsilon_1, 1]$, then $\psi_{k+1}(a+b) \leq \lambda(a+b)$. On the other hand, $\psi_{k+1}(a) \geq \lambda a$ and $\psi_{k+1}(b) \geq \lambda b$, hence $\psi_{k+1}(a+b) \leq \psi_{k+1}(a) + \psi_{k+1}(b)$.
If $a+b$ is greater than 1, then $(a+b)\text{mod } 1 < \alpha + \epsilon_1$. Let $x = \alpha + \epsilon_1 - b$ and $y = 1 - \alpha - \epsilon_1$. So $(a+b)\text{mod } 1 = a - x - y$. Then

$$\begin{aligned}
\psi_{k+1}(a)+\psi_{k+1}(b) &= \psi'_k(a)+\psi'_k(b)+\lambda a+\lambda b \\
&\geq \psi'_k(a+b)+\lambda a+\lambda b && ((a+b)\text{mod } 1<\alpha+\epsilon_1) \\
&= \psi'_k(a-x)+\lambda a+\lambda b && (a-x=(a+b)-(\alpha+\epsilon_1))
\end{aligned} \tag{7.4}$$

because ψ'_k has period $\alpha+\epsilon_1$. Also,

$$\begin{aligned}
\psi_{k+1}(a+b) &= \psi_{k+1}(a-x-y) \\
&\leq \psi_{k+1}(a-x)+\tfrac{y}{1-\alpha} && (\text{All negative slopes in } \psi_{k+1} \text{ are } -\tfrac{1}{1-\alpha}) \\
&= \psi_{k+1}(a-x)+\lambda(b+x) && (\text{by definition of } \lambda, x, y) \\
&= \psi'_k(a-x)+\lambda(a-x)+\lambda(b+x) && (\text{by Lemma 107 because } 0\leq a-x\leq\alpha+\epsilon_1) \\
&= \psi'_k(a-x)+\lambda a+\lambda b
\end{aligned} \tag{7.5}$$

From (7.4) and (7.5) we get that $\psi_{k+1}(a)+\psi_{k+1}(b)\geq\psi_{k+1}(a+b)$.

Case 2 : b is in the range $[\alpha+\epsilon_1,1]$

If $a+b$ is also in the range $[\alpha+\epsilon_1,1]$, then $\psi_{k+1}(a+b)\leq\psi_{k+1}(b)$. Therefore, $\psi_{k+1}(a+b)\leq\psi_{k+1}(a)+\psi_{k+1}(b)$.

Now we consider the case where $a+b>1$. Since every line segment with negative slope in the graph of ψ_{k+1} has slope $-\frac{1}{1-\alpha}$, then in the range $[0,a]$ the line of slope $-\frac{1}{1-\alpha}$ passing through $(a,\psi_{k+1}(a))$ lies above the graph of ψ_{k+1}. Formally, for every $x\in[0,a]$,

$$-\frac{1}{1-\alpha}x+\psi_{k+1}(a)+\frac{a}{1-\alpha}\geq\psi_{k+1}(x). \tag{7.6}$$

Now

$$\psi_{k+1}(a)+\psi_{k+1}(b)=\psi_{k+1}(a)+\frac{1-b}{1-\alpha}\geq-\frac{a+b-1}{1-\alpha}+\psi_{k+1}(a)+\frac{a}{1-\alpha}\geq\psi_{k+1}(a+b-1)$$

where the first equality is because $\psi_{k+1}(b)=\frac{1}{1-\alpha}(1-b)$, and the last inequality follows by (7.6). Therefore, we get $\psi_{k+1}(a)+\psi_{k+1}(b)\geq\psi_{k+1}(a+b-1)=\psi_{k+1}(a+b)$. □

Fact 111. *$\psi_i(x)$ is a symmetric function.*

Proof. It is straightforward to show that ψ_0 is symmetric. Notice that, by construction, the function $\psi_{i+1} - \psi_i$ satisfies

$$(\psi_{i+1} - \psi_i)(x) + (\psi_{i+1} - \psi_i)(\alpha - x) = 0.$$

Therefore, if ψ_i is symmetric, also ψ_{i+1} is symmetric. □

Theorem 112. *For $i \geq 0$, the function ψ_i is a facet.*

Proof. Since ψ_i is a function that is piecewise linear, subadditive, symmetric and has only two slopes, then, by Theorems 100 and 105, ψ_i is a facet. □

7.4 Analysis of the limit function

Recall that ψ is the function defined by

$$\psi(x) = \lim_{i \to \infty} \psi_i(x)$$

for every $x \in [0, 1[$.

Fact 106 implies the following.

Fact 113. *Let $\gamma = \alpha - \sum_{i=1}^{+\infty} 2^{i-1}\epsilon_i$. Then $\gamma > 0$ by (7.2) and $\gamma < \gamma_i$ for all i, and the value s_i of the positive slope in ψ_i is bounded above by $\frac{1-\gamma}{(1-\alpha)\gamma}$.*

We can now show the following lemma.

Lemma 114. *For any x, the sequence $\{\psi_i(x)\}_{i=1,2,3,\ldots}$ is a Cauchy sequence, and therefore it converges. Moreover, the sequence of functions $\{\psi_i\}_{i=1,2,3,\ldots}$ converges uniformly to ψ.*

Proof. By Fact 106, there are 2^i intervals where ψ_i has positive slope, each of length $\frac{\gamma_i}{2^i}$. Note that $|\psi_i(x) - \psi_{i+1}(x)| \leq s_{i+1}\frac{\gamma_i}{2^i}$ since the values of the two functions match at the ends of the positive-slope intervals of ψ_i.

By Fact 113, $s_{i+1} \leq \frac{1-\gamma}{(1-\alpha)\gamma}$ and we know that $\gamma_i < \alpha$. So $|\psi_i(x) - \psi_{i+1}(x)| \leq C\frac{1}{2^i}$ where $C = \alpha\frac{1-\gamma}{(1-\alpha)\gamma}$. Therefore, $|\psi_n(x) - \psi_m(x)| \leq \sum_{i=n}^{m-1} C\frac{1}{2^i}$ if $n < m$. We can bound this expression using

$$\sum_{i=n}^{m-1} C\frac{1}{2^i} \leq \sum_{i=n}^{\infty} C\frac{1}{2^i} = C\frac{1}{2^{n-1}}$$

This implies that the sequence is Cauchy and hence convergent. Moreover, since the bound on $|\psi_n(x) - \psi_m(x)|$ does not depend on x, the above argument immediately implies that the sequence of functions ψ_i converges uniformly to ψ. □

This also implies the following corollary.

Corollary 115. *The function ψ is continuous.*

Proof. ψ_i is continuous for each $i \in \{1, 2, 3, \ldots\}$ by construction. Since this sequence of functions converges uniformly to ψ, ψ is continuous [57]. □

For each integer $i \geq 0$, define S_i to be the subset of points of $]0, 1[$ over which the function ψ_i has negative slope. By Fact 106, S_i is the union of 2^i open intervals. Furthermore, by construction $S_i \subseteq S_{i+1}$ for every $i \in \mathbb{N}$. The set $S \subseteq [0, 1]$ defined by

$$S = \cup_{i=0}^{\infty} S_i,$$

is the set of points over which ψ has negative slope, and it is an open set since it is the union of open intervals.

Fact 116. *The set S is dense in* $[0, 1]$.

Proof. Let $a \in [0, 1]$. We need to show that, for any $\delta > 0$, there exists $b \in S$ such that $|a - b| < \delta$. Choose i such that $\frac{1}{2^i} < \delta$. If ψ_i has negative slope in a, then $a \in S$ and we are done. Thus a is in a positive slope interval of ψ_i. By Fact 106, such an interval has length $\frac{\gamma_i}{2^i}$, hence there exists a point b in a negative slope interval of ψ_i, and thus in S, such that $|a - b| \leq \frac{\gamma_i}{2^i} < \delta$, since $\gamma_i < 1$. □

Fact 117. *The function ψ is not piecewise linear.*

Proof. Suppose by contradiction that ψ is piecewise linear. Then, for some $\varepsilon > 0$, the restriction of ψ to $]0, \varepsilon[$ is linear. By Fact 116, $]0, \varepsilon[$ contains a point of S, therefore ψ has constant negative slope in $]0, \varepsilon[$. Since $\psi(0) = 0$, then $\psi(x) < 0$ for every $x \in]0, \varepsilon[$, a contradiction. □

Lemma 118. *The function ψ is a minimal valid function.*

Proof. By Theorem 100, we only need to show that ψ is subadditive and symmetric. This follows from pointwise convergence. □

We finally show that the function ψ is a facet.

Theorem 119. *The function ψ is a facet for the problem (7.1).*

Proof. We will use the Facet Theorem (Theorem 103). We show that if a minimal valid function satisfies the set of equalities $E(\psi)$, then it coincides with ψ everywhere.

Consider any minimal valid function ϕ that satisfies the set of equalities $E(\psi)$. Therefore, if $\psi(u) + \psi(v) = \psi(u + v)$, then $\phi(u) + \phi(v) = \phi(u + v)$.

We first show the following.

Claim 1. $\phi(x) = \psi(x)$ *for all* $x \in S$

For any maximal interval $]a, b[$ over which the graph of ψ has a negative slope, consider the following intervals : $U = [(a + b)/2, b]$, $V = [1 - ((b - a)/2), 1]$ and therefore $U + V = [a, b]$. It is easy to see that $\psi(u) + \psi(v) = \psi(u + v)$ for $u \in U$, $v \in V$. This implies $\phi(u) + \phi(v) = \phi(u + v)$. Now Lemma 104 (the Interval Lemma) implies that ϕ are straight lines over U, V and $U + V$.

We now use an inductive argument to prove that not only do the slopes of ψ and ϕ coincide on intervals where the slope of ψ is negative, in fact $\psi(x) = \phi(x)$ for all x in these intervals.

Every maximal segment s with negative slope in ψ also appears in ψ_i for some i (i.e. $s \subseteq S_i$ for some i). Let $index(s)$ be the least such i. We prove that $\psi(x) = \phi(x)$ for every s with negative slope by induction on $index(s)$. $\phi(\alpha) = \psi(\alpha) = 1$ and $\phi(0) = \psi(0) = 0$ since ϕ is assumed to be a valid inequality. This implies that ϕ is the same as ψ in the range $[\alpha, 1]$. This proves the base case of the induction.

By the induction hypothesis, we assume the claim is true for negative-slope segments s with $index(s) = k$. Consider all negative-slope segments s with $index(s) = k + 1$. Amongst these consider the segment s_c which is closest to the origin. Let the midpoint of this segment be m. We know that $2m$ is the start of a negative-slope segment s' in ψ with $index(s') = k$. By construction, $\psi(m) + \psi(m) = \psi(2m)$. So $\phi(m) + \phi(m) = \phi(2m)$. From the induction hypothesis, we know that $\psi(2m) = \phi(2m)$ and so $\phi(m) = \frac{1}{2}\phi(2m) = \frac{1}{2}\psi(2m) = \psi(m)$.

Now consider any other negative-slope segment s with $index(s) = k + 1$ and let its midpoint be m_s. Note that $m_s + m$ is the start of a negative-slope segment s'

with $index(s') = k$. So

$$\phi(m_s + m) = \psi(m_s + m) \tag{7.7}$$

because of the inductive hypothesis. Note that $\psi(m_s + m) = \psi(m_s) + \psi(m)$ by construction. So, $\phi(m_s + m) = \phi(m_s) + \phi(m)$. Since we showed that $\phi(m) = \psi(m)$, (7.7) implies that $\phi(m_s) = \psi(m_s)$. Since the values coincide at the midpoints of these segments and the slopes of the segments are the same, $\phi(x) = \psi(x)$ for any x in the domain of these segments. This concludes the proof of Claim 1.

We now use Claim 1 to show that

$$\phi(x) \leq \psi(x) \text{ for all } x \in [0, 1[. \tag{7.8}$$

By Lemma 118 ψ is a minimal valid function and ϕ is assumed to be minimal, thus symmetry combined with (7.8) would imply that $\phi(x) = \psi(x)$ for all $x \in [0, 1[$.

Claim 2. Let $\bar{x} \in]0, 1[$. For every positive integer n, there exists $y_n, z_n \in S$ such that $0 < z_n < \frac{1}{n}$ and $y_n + z_n = \bar{x}$.

Since S is open and, by Fact 116, it is dense in $[0, 1]$, for any positive integer n there exists an open interval $I \subseteq S$ such that $0 < \bar{x} - y < \frac{1}{n}$ for all $y \in I$. Let $I =]u_1, u_2[$ with $u_1 \neq u_2$. Also, since S is dense in $[0, 1]$, there exists $z \in S \cap]\bar{x} - u_2, \bar{x} - u_1[$. Let this z be z_n and $y_n = \bar{x} - z \in I$. Since $I \subseteq S$, we have that $y_n \in S$. This concludes the proof of Claim 2.

Note that the sequence $\{y_n\}$ converges to $\bar{x}$ and $\{z_n\}$ converges to 0. For every positive integer n we have

$$\begin{array}{rrcl} & \phi(\bar{x}) & = & \phi(y_n + z_n) \\ & & \leq & \phi(y_n) + \phi(z_n) \qquad \text{(By subadditivity of } \phi) \\ & & = & \psi(y_n) + \psi(z_n) \\ \Rightarrow & \phi(\bar{x}) & \leq & \lim_{n\to\infty}(\psi(y_n) + \psi(z_n)) \\ & & = & \psi(\bar{x}) \qquad \text{(By continuity of } \psi) \end{array}$$

□

Bibliography

[1] COmputational INfrastructure for Operations Research (COIN-OR) http://www.coin-or.org.

[2] http://wpweb2.tepper.cmu.edu/fmargot/.

[3] R statistical software, http://www.r-project.org/.

[4] http://www.andrew.cmu.edu/user/anureets/osc/osc.htm.

[5] J. Aczél. *Lectures on functional equations and their applications*. Academic Press, New York, 1966.

[6] K. Andersen, Q. Louveaux, and R. Weismantel. An analysis of mixed integer linear sets based on lattice point free convex sets. Technical report, manuscript, 2007.

[7] K. Andersen, Q. Louveaux, and R. Weismantel. Integral farkas type lemmas for systems with equalities and inequalities. CORE Discussion Papers 2007051, Universit catholique de Louvain, Center for Operations Research and Econometrics (CORE), July 2007.

[8] K. Andersen, Q. Louveaux, R. Weismantel, and L. Wolsey. Cutting planes from two rows of a simplex tableau. In *Proceedings of Integer Programming and Combinatorial Optimization XII*, pages 1–15, 2007.

[9] K. Andersen, C. Wagner, and R. Weismantel. On an analysis of the strength of mixed integer cutting planes from multiple simplex tableau rows. Technical report, manuscript, December 2008.

[10] E. Balas. Personal Communication.

[11] E. Balas. Intersection cuts - a new type of cutting planes for integer programming. *Operations Research*, 19:19–39, 1971.

[12] E. Balas. Disjunctive programming : properties of the convex hull of feasible points. Technical report, GSIA Management Science Research Report MSRR 348, Carnegie Mellon University, 1974.

[13] E. Balas. Disjunctive programming. *Annals of Discrete Mathematics*, 5:3–51, 1979.

[14] E. Balas, S. Ceria, and G. Cornuéjols. A lift-and-project cutting plane algorithm for mixed 0-1 programs. *Mathematical Programming*, 58:295–324, 1993.

[15] E. Balas, S. Ceria, G. Cornuéjols, and N. Natraj. Gomory cuts revisited. *Operations Research Letters*, 19:1–9, 1996.

[16] E. Balas and A. Saxena. Optimizing over the split closure. *Mathematical Programming, Series A*, 113:219–240, 2008.

[17] A. Barvinok. *A Course in Convexity*, volume 54 of *Graduate Studies in Mathematics*. American Mathematical Society, 2002.

[18] A. Basu, P. Bonami, G. Cornuéjols, and F. Margot. On the relative strength of split, triangle and quadrilateral cuts. *Mathematical Programming Series A*, 89:1–44, 1998.

[19] A. Basu, M. Campelo, M. Conforti, G. Cornuejols, and G. Zambelli. On lifting integer variables in minimal inequalities. In *Proceedings of Integer Programming and Combinatorial Optimization (IPCO) XV*, 2010.

[20] A. Basu, M. Conforti, G. Cornuejols, and G. Zambelli. A counterexample to a conjecture by gomory and johnson. Technical report, manuscript, November 2009.

[21] A. Basu, M. Conforti, G. Cornuéjols, and G. Zambelli. Maximal lattice-free convex sets in linear subspaces. Technical report, manuscript, 2009.

[22] A. Basu, M. Conforti, G. Cornuéjols, and G. Zambelli. Minimal inequalities for an infinite relaxation of integer programs. to appear in SIAM Journal of Discrete Mathematics, 2010.

[23] R. E. Bixby, Z. Gu, E. Rothberg, and R. Wunderling. Mixed-integer programming : A progress report. pages 309–326, 2004.

[24] R.E. Bixby, S. Ceria, C.M. McZeal, and M.W.P. Savelsbergh. Miplib 3.0. `http://www.caam.rice.edu/~bixby/miplib/miplib.html`.

[25] V. Borozan and G. Cornuéjols. Minimal valid inequalities for integer constraints. *Mathematics of Operations Research*, 34:538–546, 2009.

[26] A. Caprara and A.N. Letchford. On the separation of split cuts and related inequalities. *Mathematical Programming*, 94:279–294, 2003.

[27] M. Conforti, G. Cornuejols, and G. Zambelli. Equivalence between intersection cuts and the corner polyhedron. 2009.

[28] M. Conforti, G. Cornuejols, and G. Zambelli. A geometric perspective on lifting. Technical report, manuscript, 2009.

[29] W.J. Conover. *Practical Nonparametric Statistics*. Wiley, 3rd edition, 1999.

[30] J. B. Conway. *A Course in Functional Analysis*. Graduate Texts in Mathematics, Springer, Berlin, 1990.

[31] W. Cook, R. Kannan, and A. Schrijver. Chvátal closures for mixed integer programming problems. *Mathematical Programming*, 47:155–174, 1990.

[32] G. Cornuéjols. Revival of the gomory cuts in the 1990's. *Annals of Operations Research*, 149:63–68, 2007.

[33] G. Cornuéjols and Y. Li. A connection between cutting plane theory and the geometry of numbers. *Mathematical Programming*, 93:123–127, 2002.

[34] G. Cornuéjols and F. Margot. On the facets of mixed integer programs with two integer variables and two constraints. *Mathematical Programming, Series A*, 120:429–456, 2009.

[35] S. Dash, O. Gunluk, and A. Lodi. On the mir closure of polyhedra. In *Proceedings of Integer Programming and Combinatorial Optimization (IPCO) XII*, volume 4513 of *Lecture Notes in Computer Science*, pages 337–351, 2008.

[36] S.S. Dey, J.-P. P. Richard, Y. Li, and L. A. Miller. On the extreme inequalities of infinite group problems. *Mathematical Programming*, 121:145–170, 2010.

[37] S.S. Dey and L.A. Wolsey. Lifting integer variables in minimal inequalities corresponding to lattice-free triangles. In *Proceedings of Integer Programming and Combinatorial Optimization (IPCO) XIII*, volume 5035 of *Lecture Notes in Computer Science*, pages 463–475, 2008.

[38] S.S. Dey and L.A. Wolsey. Constrained infinite group relaxations of mips. *manuscript*, 2009.

[39] J. P. Doignon. Convexity in crystallographic lattices. *Journal of Geometry*, 3:71–85, 1973.

[40] D. Espinoza. Computing with multi-row gomory cuts. In *Proceedings of Integer Programming and Combinatorial Optimization (IPCO) XIII*, volume 5035 of *Lecture Notes in Computer Science*, pages 214–224, 2008.

[41] M.X. Goemans. Worst-case comparison of valid inequalities for the tsp. *Mathematical Programming*, 69:335–349, 1995.

[42] R.E. Gomory. Outline of an algorithm for integer solutions to linear programs. *Bulletin of the American Mathematical Society*, 64:275–278, 1958.

[43] R.E. Gomory. Some polyhedra related to combinatorial problems. *Linear Algebra and its Applications*, 2:451–558, 1969.

[44] R.E. Gomory and E.L. Johnson. Some continuous functions related to corner polyhedra (part 1). *Mathematical Programming*, 3:23–85, 1972.

[45] R.E. Gomory and E.L. Johnson. Some continuous functions related to corner polyhedra (part 2). *Mathematical Programming*, 3:359–389, 1972.

[46] R.E. Gomory and E.L. Johnson. T-space and cutting planes. *Mathematical Programming Series B*, 96:341–375, 2003.

[47] J-B Hiriart-Urruty and C. Lemaréchal. *Fundamentals of Convex Analysis*. Springer, 2001.

[48] E.L. Johnson. On the group problem for mixed integer programming. *Mathematical Programming Study*, 2:137–179, 1974.

[49] E.L. Johnson. Characterization of facets for multiple right-hand side choice linear programs. *Mathematical Programming Study*, 14:112–142, 1981.

[50] A.H. Lenstra Jr. Integer programming with a fixed number of variables. *Mathematics of Operations Research*, 8:538–548, 1983.

[51] K. Kianfar and Y. Fathi. Generalized mixed integer rounding inequalities: facets for infinite group polyhedra. *Mathematical Programming*, 120:313–346, 2009.

[52] L. Lovász. Geometry of numbers and integer programming. In M. Iri and K. Tanabe, editors, *Mathematical Programming: Recent Developements and Applications*, pages 177–201. Kluwer, 1989.

[53] F. Margot. Testing cut generators for mixed integer linear programming. *Mathematical Programming Computation*, 1:69–95, 2009.

[54] R.R. Meyer. On the existence of optimal solutions to integer and mixed-integer programming problems. *Mathematical Programming*, 7:223–235, 1974.

[55] G.L. Nemhauser and L.A. Wolsey. A recursive procedure to generate all cuts for 0-1 mixed integer programs. *Mathematical Programming*, 46:379–390, 1990.

[56] R.T. Rockafellar. *Convex Analysis*. Princeton University Press, 1969.

[57] W. Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, 3rd edition, 1976.

[58] A. Schrijver. *Theory of Linear and Integer Programming*. John Wiley and Sons Ltd., 1986.

[59] S. Straszewicz. Uber exponierte punkte abgeschlossener punktemengen. *Fund. Math.*, 24:139–143, 1935.

[60] G. Zambelli. On degenerate multi-row gomory cuts. *Operations Research Letters*, 37:21–22, 2009.