Supplemental Appendix to "Ambiguity Shifts and the 2007–2008 Financial Crisis"

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1 Reference Model Estimation

I estimate the time series parameters of the reference model using a Gibbs sampling procedure. Recall the Gibbs sampling allows to sequentially make parameter draws from conditional posteriors. Because the model can be broken down into as many conditional posteriors as needed, it is possible to fully estimate the reference model specified in Section **??**.

Recall that for the model of Section **??**, the observations in the economy are given by:

$$\begin{aligned} y_{it} &= \xi_{i,s_t} + \rho_i \xi_{I+1,s_t} + u_{it}, \qquad i = 1, \ldots, I \\ y_{ct} &= \xi_{I+1,s_t} + u_{ct}, \end{aligned}$$

where s_t is the indicator of the state at date t. In reality, observations of book value occur at a quarterly frequency while observations of the aggregate signal occur at a monthly frequency. Denote by n_y the number of periods in between observations of firm-specific signals and by t_y the number of available observations of the firm-specific signals.

To reduce the number of parameters to be estimated, I impose additional restrictions on the model. In particular, I assume that the vector of the firm-specific components $z_{ft} \equiv [z_{1t}, \ldots, z_{It}]$ of the hidden state vector z evolves independently of the aggregate component, z_{ct}. That is, I assume that z_{ft} and z_{ct} evolve as two independent Markov chains. The vector of firm-specific components z_f evolves as an n_f-state Markov chain, with values $\zeta_{f1}, \ldots, \zeta_{f,n_f}$ and transition probability matrix Ω_f defined by:

$$\{\Omega_f\}_{jk} \equiv \omega_{f,ik} = \mathbb{P}\left(z_{f,t+1} = \zeta_{fk} \middle| z_{ft} = \zeta_{fj} \right).$$

Similarly, the aggregate component z_{ct} evolves as an n_c Markov chain, with values $\zeta_{c1}, \ldots, \zeta_{cn_c}$ and transition matrix Ω_c defined by:

$$\{\Omega_c\}_{jk} \equiv \omega_{ci,jk} = \mathbb{P}\left(z_{c,t+1} = \zeta_{ck} \middle| z_{ct} = \zeta_{cj} \right).$$

I impose also the assumption that the signal errors of the aggregate signal are uncorrelated with the signal errors of the firm-specific signals but allow for the errors of the firm-specific signals to be cross-sectionally correlated. That is, I partition the signal covariance matrix into:

$$\Sigma_u = \begin{bmatrix} \Sigma_{uf} & \vec{0}_{I,1} \\ \vec{0}_{1,I} & \Sigma_{uc} \end{bmatrix},$$

where Σ_{uf} is the covariance matrix of the firm-specific signals and Σ_{uc} is the variance of the aggregate signal. Notice that this formulation allows me to estimate the firm value at default directly from the signal observations: since z_{ft} and z_{ct} evolve as two independent Markov chains, it is possible to recover the lowest value of the firm-specific component and the lowest value of the common component without observing default.

Denote by Θ the full set of parameters to be estimated:

$$\Theta = \left\{ \Omega_f, \Omega_c, \zeta_{f1}, \ldots, \zeta_{fn_f}, \zeta_{c1}, \ldots, \zeta_{cn_c}, \rho_1, \ldots, \rho_I, \Sigma_{uf}, \Sigma_{uc}, \{s_{ft}\}_{t=1}^{t_y}, \{s_{ct}\}_{t=1}^{T} \right\}$$

and by Θ_{-A} the set of all parameters except A: $\Theta_{-A} = \Theta \backslash A$. With the above assumptions, the main steps in the Gibbs procedure are as follows:

Step 1. Conditional on a draw of $\Theta_{-\Omega_f}$, make a draw of Ω_f

Step 2. Conditional on a draw of $\Theta_{-\Omega_c}$, make a draw of Ω_c

Step 3. Conditional on a draw of $\Theta_{-\{\zeta_{ci}\}_{i=1}^{n_c}}$, make a draw of $\zeta_{c1}, \ldots, \zeta_{c,n_c}$

Step 4. Conditional on a draw of $\Theta_{-\left\{\zeta_{fi}\right\}_{i=1}^{n_f}}$, make a draw of $\zeta_{f1}, \ldots, \zeta_{f,n_f}$

Step 5. Conditional on a draw of $\Theta_{-\{\rho_i\}_{i=1}^{I}}$, make a draw of $\rho_1, \ldots, \rho_I$

Step 6. Conditional on a draw of $\Theta_{-\Sigma_{uf}}$, make a draw of Σ_{uf}

Step 7. Conditional on a draw of $\Theta_{-\Sigma_{uc}}$, make a draw of Σ_{uc}

Step 8. Conditional on a draw of $\Theta_{-\{s_{ct}\}_{t=1}^{T}}$, make a draw of $\{s_{ct}\}_{t=1}^{T}$

Step 9. Conditional on a draw of $\Theta_{-\left\{s_{ft}\right\}_{t=1}^{t_y}}$, make a draw of $\{s_{ft}\}_{t=1}^{t_y}$

Step 10. Permute the state indicators

For the conditional posteriors below, I rely on Gibbs sampling results for regime-switching models. The initial application of MCMC estimation methods to regime-switching models in due to Albert and Chib (1993), who estimate an autoregressive model with Markov jumps following a two-state Markov process. McCulloch and Tsay (1994) extend this to situations where the regime-switching model includes non-regime-specific (common) components. For the most part (and in the algorithm below), practical MCMC

estimation uses the principle of data augmentation and treats the indicators of the state of the latent Markov chain as missing data. Treating observations of the latent Markov chain as missing data allows for the use of conjugate priors in estimating the parameters of the model. For a more exhaustive discussion of the use of MCMC methods for estimating the parameters of Markov chains, see Fruhwirth-Schnatter (2006).

1.1 Conditional on a draw of $\Theta_{-\Omega_f}$, make a draw of Ω_f

Denote by $\tilde{\Omega}_f$ the n_y-periods-ahead transition probability matrix of the firm-specific value vector z_{ft}: $\tilde{\Omega}_f = \Omega_f^{n_y}$. Since obsevations of the firm-specific signals occur only every n_y periods and the aggregate signals are not informative about the firm-specific state, I make draws of $\tilde{\Omega}_f$ and then infer the corresponding draw of Ω_f. Generalizing the results of Albert and Chib (1993) and McCulloch and Tsay (1994) to the multiple state case, the conjugate prior for the jth row of $\tilde{\Omega}_f$ is :

$$\tilde{\omega}_{f,j} \sim Dir\left(\alpha_{j1}^f, \ldots, \alpha_{j,n_f}^f\right),$$

where Dir denotes the Dirichlet distribution[1]. The posterior is then given by:

$$\tilde{\omega}_{f,j} \sim Dir\left(\alpha_{j1}^f + n_{j1}^f, \ldots, \alpha_{j,n_f}^f + n_{jn_f}^f\right),$$

where n_{jk}^f is the number of times the Markov chain z_f transitions from state j to state k in the current draw of $\{s_{ft}\}_{t=1}^{t_y}$. Once a draw of $\tilde{\Omega}_f$ is made, the corresponding draw of the original transition probability matrix is computed as $\Omega_f = \tilde{\Omega}_f^{\frac{1}{n_y}}$.

1.2 Conditional on a draw of $\Theta_{-\Omega_c}$, make a draw of Ω_c

Similarly to $\tilde{\Omega}_f$, the conjugate prior for the jth row of Ω_c is the Dirichlet distribution:

$$\omega_{c,j} \sim Dir\left(\alpha_{j1}^c, \ldots, \alpha_{j,n_c}^c\right),$$

and the posterior is given by:

$$\omega_{c,j} \sim Dir\left(\alpha_{j1}^c + n_{j1}^c, \ldots, \alpha_{j,n_c}^c + n_{j,n_c}^c\right),$$

where n_{jk}^c is the number of times the Markov chain z_c transitions from state j to state k in the current draw of $\{s_{ct}\}_{t=1}^T$.

[1]Recall that the Dirichlet distribution generalizes the beta distribution to the multinomial case

1.3 Conditional on a draw of $\Theta_{-\{\zeta_{ci}\}_{i=1}^{n_c}}$, make a draw of $\zeta_{c1}, \ldots, \zeta_{c,n_c}$

To derive the conditional posterior of ζ_{ci}, $i = 1, \ldots, n_c$, notice that the firm-specific signals contain information about the common component of the fundamental asset values. In particular, notice that the likelihood function of the signals is given by:

$$\mathcal{L}\left(y|\,\Theta\right) \;\propto\; \exp\left\{-\frac{1}{2}\sum_{\tau=1}^{t_y}\hat{y}'_{f,\tau\Delta_y}\Sigma_{uf}^{-1}\hat{y}_{f,\tau\Delta_y} - \frac{1}{2}\sum_{t=1}^{T}\frac{(y_{ct}-\zeta_{c,s_{ct}}-\overline{u}_c)^2}{\Sigma_{uc}}\right\},$$

where $\hat{y}_{f,\tau\Delta_y} = y_{f,\tau\Delta_y} - \zeta_{f,s_{f,\tau\Delta_y}} - \rho\zeta_{c,s_{c,\tau\Delta_y}} - \overline{u}_f$. Let $i_1 < i_2 < \ldots < i_{n_j}$ denote all the time indices such that $s_{ci_k} = j$ and let $y_{c,j} = \left(y_{ci_1}, \ldots, y_{ci_{n_j}}\right)'$. The conjugate prior is given by:

$$\zeta_{cj} \sim N\left(\zeta_{c,0j},\ \Sigma_{uc}A_{c,0j}^{-1}\right)$$

and the conditional prior by:

$$\zeta_{cj} \sim N\left(\bar{\zeta}_{cj},\ \Sigma_{uc}\bar{A}_{cj}^{-1}\right),$$

where

$$\begin{aligned}
\bar{A}_{cj} &= n_{y_j}\Sigma_{uc}\rho'\Sigma_{uf}^{-1}\rho + n_j + A_{c,0j} \\
\bar{\zeta}_{cj} &= \bar{A}_{cj}^{-1}\left[\Sigma_{uc}\rho'\Sigma_{uf}^{-1}\sum_{k=1}^{n_{y_j}}(y_{f,\tau_k\Delta_y} - \zeta_{f,s_{1\tau_k\Delta_y}} - \bar{u}_f) + \sum_{k=1}^{n_j}(y_{c,i_k} - \overline{u}_c) + A_{c,0j}\zeta_{c,0j}\right]
\end{aligned}$$

1.4 Conditional on a draw of $\Theta_{-\{\zeta_{fi}\}_{i=1}^{n_f}}$, make a draw of $\zeta_{f1}, \ldots, \zeta_{f,n_f}$

Denote: $\tilde{y}_{ft} = y_{ft} - \rho\zeta_{cs_{ct}} - \bar{u}$. Let $j_1 < j_2 < \ldots < j_{n_j}$ denote all the time indices such that $s_{fj_k} = j$ and let $\tilde{y}_{fj} = \left(\tilde{y}_{f,j_1}, \ldots, \tilde{y}_{f,j_{n_j}}\right)'$. Then the conjugate prior distribution is:

$$\zeta_{fj} \sim N\left(\zeta_{f,0j},\ \Sigma_{uf}A_{f,0j}^{-1}\right)$$

and the conditional posterior by:

$$\zeta_{fj} \sim N\left(\bar{\zeta}_{fj},\ \Sigma_{uf}\bar{A}_{fj}^{-1}\right),$$

where:

$$\begin{aligned}
\bar{A}_{fj} &= n_j + A_{f,j0} \\
\bar{\zeta}_{fj} &= \bar{A}_{fj}^{-1}\left[\sum_{k=1}^{n_j}\tilde{y}_{f,j_k} + A_{f,j0}\zeta_{f,j0}\right]
\end{aligned}$$

1.5 Conditional on a draw of $\Theta_{-\{\rho_i\}_{i=1}^I,\Sigma_{uf}}$, make a draw of $\rho_1,\ldots,\rho_I$ and

Define: $\tilde{y}_{ft} = y_{ft} - \overline{u}_f - \zeta_{fs_{ft}}$. The prior distribution for the vector ρ is then

$$\rho \sim N\left(\beta_0, \Sigma_{uf} A_{0\rho}^{-1}\right)$$

and the conjugate posterior is given by:

$$\rho \sim N\left(\beta_*, \Sigma_{uf} A_{*,\rho}^{-1}\right),$$

where:

$$\begin{aligned} A_{*,\rho} &= \sum_{t=1}^{t_y} \zeta_{c,s_{ct}}^2 + A_{0\rho} \\ \beta_* &= A_{*,\rho}^{-1}\left(\sum_{t=1}^{t_y} \tilde{y}_{ft}\zeta_{c,s_{ct}} + A_{0\rho}\beta_0\right). \end{aligned}$$

The conjugate prior for Σ_{uf} is given by:

$$\Sigma_{uf} \sim IW(\nu, V)$$

and the posterior is given by:

$$\Sigma_{uf} \sim IW(\nu + t_y, V + S),$$

where

$$S = \sum_{t=1}^{t_y}\left(\tilde{y}_{f,t\Delta_y} - \rho\zeta_{c,s_{c,t\Delta_y}}\right)'\left(\tilde{y}_{f,t\Delta_y} - \rho\zeta_{c,s_{c,t\Delta_y}}\right) + (\beta_* - \beta_0)' A_{0\rho}(\beta_* - \beta_0).$$

1.6 Conditional on a draw of $\Theta_{-\Sigma_{uc}}$, make a draw of Σ_{uc}

In this case, the conjugate prior is an inverse χ^2-distribution:

$$\Sigma_{uc} \sim \frac{\nu_c \bar{s}_c^2}{\chi^2(\nu_c)}$$

and the conditional posterior by:

$$\Sigma_{uc} \sim \frac{\nu_c \bar{s}_c^2 + TE_c}{\chi^2(\nu_c + T)},$$

where:

$$TE_c = \sum_{t=1}^{T} (y_{ct} - \zeta_{cs_{ct}} - \overline{u}_c)^2.$$

1.7 Conditional on a draw of $\Theta_{-\{s_{ct}\}_{t=1}^T}$, make a draw of $\{s_{ct}\}_{t=1}^T$

Denote by S_{ct} the history of observations of the aggregate regime up to date t, $S_{c,-t}$ the full history of the aggregate regime except at date t. The conditional posterior at dates when only the aggregate signal is observed is then given by:

$$\mathbb{P}\left(s_{ct} | \{y_{c\tau}\}_{\tau=1}^T, S_{c,-t}\right) \propto \mathbb{P}(s_{ct}|s_{c,t-1})\mathbb{P}(s_{c,t+1}|s_{ct}) \exp\left\{-\frac{1}{2}\frac{(y_{ct} - \zeta_{c,s_{ct}})^2}{\Sigma_{uc}}\right\},$$

and, on dates when firm-specific signals are also observed, by:

$$\begin{aligned}\mathbb{P}\left(s_{ct} | \{y_{c\tau}\}_{\tau=1}^T, S_{c,-t}\right) \propto\ & \mathbb{P}(s_{ct}|s_{c,t-1})\mathbb{P}(s_{c,t+1}|s_{ct}) \exp\left\{-\frac{1}{2}\frac{(y_{ct} - \zeta_{c,s_{ct}})^2}{\Sigma_{uc}}\right\} \\ & \times \exp\left\{-\frac{1}{2}\left(y_{ft} - \zeta_{f,s_{ft}} - \rho\zeta_{cs_{ct}}\right)' \Sigma_{uf}^{-1} \left(y_{ft} - \zeta_{f,s_{ft}} - \rho\zeta_{cs_{ct}}\right)\right\}\end{aligned}$$

1.8 Conditional on a draw of $\Theta_{-\{s_{ft}\}_{t=1}^{t_y}}$, make a draw of $\{s_{ft}\}_{t=1}^{t_y}$

Denote S_{ft} the history of observations of the firm-specific regime up to date t, $S_{f,-t}$ the full history of observations of the firm-specific regime except at date t. Then, the conditional posterior for s_{ft} is given by:

$$\begin{aligned}\mathbb{P}\left(s_{ft} | \{y_{f\tau}\}_{\tau=1}^T, S_{f,-t}\right) \propto\ & \mathbb{P}(s_{ft}|s_{f,t-1})\mathbb{P}(s_{f,t+1}|s_{ft}) \\ & \times \exp\left\{-\frac{1}{2}\left(y_{ft} - \zeta_{f,s_{ft}} - \rho\zeta_{cs_{ct}}\right)' \Sigma_{uf}^{-1} \left(y_{ft} - \zeta_{f,s_{ft}} - \rho\zeta_{cs_{ct}}\right)\right\}.\end{aligned}$$

1.9 Permute the state indicators

As discussed in Fruhwirth-Schnatter (2001), the behavior of the sampler described above is somewhat unpredictable, and the sampler might be trapped at one modal region of the Markov mixture posterior distribution or may jump occasionally between different model regions causing label switching. A simple but efficient solution to obtain a sampler that explores the full Markov mixture posterior distribution is suggested in Fruhwirth-Schnatter (2006). Each draw from the Gibbs sampler is concluded by selecting randomly one of $n_f!$ possible permutations of the current labeling of the firm-specific states and one $n_c!$ possible permutations of the current labeling of the aggregate states. This permutation is then applied to the transition probability matrices Ω_f and Ω_c, the state-specific

parameters ζ_f and ζ_c and the state indicators $s_f^{t_y}$ and s_c^T.

References

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