On the Relationship Between Determinate and MSV Solutions in Linear RE Models

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ABSTRACT: In univariate RE models of the form $y_t = AE_ty_{t+1} + Cy_{t-1} + u_t$, every determinate solution is also the unique minimum state variable (MSV) solution. More generally, however, there are multivariate models of this form that have unique non-explosive solutions that differ from their MSV solution.

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1. Introduction

Much recent research in economics has emphasized the concept of determinacy of rational expectation solutions—i.e., the property of a solution being the only non-explosive solution. It is well known that in linear rational expectations (RE) models a necessary and sufficient condition for determinacy is that the number of eigenvalues of the system's matrix pencil that exceed 1.0 in modulus equals the number of non-predetermined endogenous variables.[1] In various prominent cases, this condition does not obtain so there is no unique non-explosive solution.

Some researchers[2] have focused attention on the minimum state variable (MSV) solution, defined and promoted in McCallum (1983, 1999), which is by construction unique but possibly explosive, and exists if the model has any real (non-imaginary) solution.[3] It is obviously the case that some MSV solutions are not determinate, but it is not obvious whether there are models in which a determinate solution exists but is not the MSV solution. That possibility has been hinted at by McCallum (1983, 1998) and Uhlig (1999), but examples have not been examined. There are some reasons, perhaps, to suspect that it might be true that all unique stable solutions are MSV solutions. Such a situation is easily seen to prevail in univariate models of the form $y_t = AE_t y_{t+1} + Cy_{t-1} + u_t$ and also holds for multivariate versions if the A and C matrices commute and a regularity condition due to Binder and Pesaran (1995, p. 157) obtains. Furthermore, there are recent results by Gauthier (2003) and Desgranges and Gauthier (2003) showing, among other things, that the same result holds in univariate perfect foresight models with additional

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[1] See Blanchard and Kahn (1980), Binder and Pesaran (1995), King and Watson (1998), among others.

[2] For example, Barro (1989) and Faust and Svensson (2001).

[3] It is important to note that the term "minimum state variable" is here being used in the manner of McCallum (1983, 1999) or Evans (1986), rather than that of Evans and Honkapohja (2001) or Gauthier (2003), which permits more than one MSV solution. See Section 2 below.

lagged terms and/or expected future values.

It transpires, nevertheless, that there can exist unique stable solutions that differ from the MSV solution. This is demonstrated below, in Section 3, after Section 2 outlines the specification to be utilized and provides preliminary results.

2. Preliminaries

Because our main result consists of a counterexample, it will not be necessary to utilize a framework with full generality. Instead, it will be convenient to consider the specification treated by McCallum (1983, pp. 164-166). With y_t denoting a m×1 vector of endogenous variables, the system is

$$(1) \qquad y_t = A\, E_t y_{t+1} + C\, y_{t-1} + u_t,$$

where $u_t = R\, u_{t-1} + \varepsilon_t$, with R a stable m×m matrix and ε_t a white noise vector.[4] Also, it is assumed that A is nonsingular. That is a strong assumption, which renders the formulation (1) highly inconvenient from a practical perspective, but is acceptable for the purposes at hand. In this setting, the MSV solution will be of the form

$$(2) \qquad y_t = \Omega\, y_{t-1} + \Gamma\, u_t.$$

Accordingly, $E_t y_{t+1} = \Omega(\Omega y_{t-1} + \Gamma u_t) + \Gamma R u_t$ and straightforward undetermined-coefficient reasoning yields the requirement that the solution for Ω satisfies

$$(3) \qquad A\Omega^2 - \Omega + C = 0,$$

where all of the matrices are of order m×m. There are other implications, of course, but the occurrence of multiple solutions arises entirely because of the nonlinear nature of (3); for a given Ω, Γ is determined uniquely. In this setting, the MSV concept requires that $\Omega = 0$ if $C = 0$, since otherwise the solution would in that case include extraneous

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[4] In (1), constant terms are suppressed for notational simplicity while A and C are of dimension m×m.

variables, and the MSV solution is defined generally as the one whose expression for Ω approaches 0 as C approaches a zero matrix.

With A invertible, the matrix quadratic (3) can be expressed as

$$(4)\qquad \begin{bmatrix}\Omega \\ \Omega^2\end{bmatrix} = \begin{bmatrix}0 & I \\ -A^{-1}C & A^{-1}\end{bmatrix}\begin{bmatrix}I \\ \Omega\end{bmatrix}.$$

Let M denote the 2m×2m matrix in (4) and assume, without significant loss of generality, that it is diagonalizable. Then $M = P^{-1}\Lambda P$, where Λ is diagonal with the eigenvalues of M on its diagonal. Then if P^{-1} includes the eigenvectors, we can premultiply by P to get

$$(5)\qquad \begin{bmatrix}P_{11} & P_{12} \\ P_{21} & P_{22}\end{bmatrix}\begin{bmatrix}\Omega \\ \Omega^2\end{bmatrix} = \begin{bmatrix}\Lambda_1 & 0 \\ 0 & \Lambda_2\end{bmatrix}\begin{bmatrix}P_{11} & P_{12} \\ P_{21} & P_{22}\end{bmatrix}\begin{bmatrix}I \\ \Omega\end{bmatrix},$$

where the P_{ij} are submatrices of P and where Λ_1 and Λ_2 contain the eigenvalues of M.

To obtain the MSV solution, McCallum (1983) arranges the eigenvalues (and associated eigenvectors) so that Λ_1 includes those that approach 0 as C approaches 0.[5] Then the MSV expression for Ω is implied by the second row of (5) to be[6]

$$(6)\qquad \Omega = -\,P_{22}^{-1}\,P_{21}.$$

Further, since $PM = \Lambda P$, we have (from the lower left submatrix) that $-P_{22}A^{-1}C = \Lambda_2 P_{21}$ so if the inverse of Λ_2 exists, (6) gives the solution

$$(7)\qquad \Omega = P_{22}^{-1}\Lambda_2^{-1}\,P_{22}A^{-1}C,$$

for which Ω approaches 0 as C approaches 0. For this conclusion, it needs to be true that Λ_2^{-1} exists in the limit. But the eigenvalues of M are obtained from $\det[M - \lambda I] = 0$, and using a result on the determinant of a partitioned matrix, we have that

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[5] The specification of this grouping is based on the continuity of eigenvalues with respect to the elements of the underlying matrix (M, in this case). We let C approach a zero matrix by replacing C with αC in all relevant expressions and letting the real scalar α vary continuously from 1.0 to 0.

[6] This row can be written as $(P_{21} + P_{22}\Omega)\Omega = \Lambda_2(P_{21} + P_{22}\Omega)$.

(8) $\det[M - \lambda I] = \det[A^{-1} - \lambda I] \det[-\lambda I + I(A^{-1} - \lambda I)^{-1}A^{-1}C]$.

From (8) we see that for any arrangement of the eigenvalues, half of them will approach zero and the other half will approach the eigenvalues of A^{-1} as C goes to 0. Thus with the MSV arrangement, the eigenvalues of Λ_2 approach those of A^{-1}, which are all non-zero.

It should be emphasized that (7) gives different solutions for different groupings of eigenvalues into Λ_1 and Λ_2. Since M is 2m×2m, there are $(2m)!/(m!)^2$ different groupings, each of which provides a solution. There is only one for which (7) is well defined in the limit as C approaches 0, however, since (8) implies that all others feature Λ_2 matrices that are not invertible when $C = 0$.

Now consider the particular solution given by (6) when the eigenvalues are instead arranged so that Λ_1 includes those that are smallest (in modulus). Defining $H \equiv P^{-1}$, we have $P_{21}H_{11} + P_{22}H_{21} = 0$ and from the upper left-hand submatrix of $MH = H\Lambda$ we have that $H_{21} = H_{11}\Lambda_1$. Therefore $\Omega = -P_{22}^{-1}P_{21} = P_{22}^{-1}P_{22}H_{21}H_{11}^{-1} = H_{21}H_{11}^{-1} = H_{11}\Lambda_1 H_{11}^{-1}$. But the latter has the same eigenvalues as Λ_1, which under present assumptions are the m smallest eigenvalues of M. If there is a unique stable solution, it will feature an Ω whose eigenvalues are the m smallest. Thus, if there is a unique stable RE solution, it will be given by (7) with Λ_1 including the smallest eigenvalues of M.

Is it likely that the unique stable solution and the MSV solution will coincide, if the former exists? Clearly, if the entries in C are all small, so that C is close to a zero matrix, they will coincide since the MSV solution for Ω will have near-zero eigenvalues—and these will then tend to be the smallest of M’s eigenvalues, which are those that appear in Λ_1 for the unique stable solution. Thus there is a distinct tendency for unique stable and MSV solutions to coincide. Indeed, they must coincide unless the set of eigenvalues, that includes only the m smallest, changes in composition as α goes from 1 to 0. For if it does not, then (7) will apply to the unique stable solution in the limit, making it correspond to the MSV solution.

3. Examples

We now turn to a numerical specification that provides a counterexample to the conjecture that all unique stable solutions are also MSV solutions. It is given by (1) with:

$$(9) \qquad A = \begin{bmatrix} -1.5 & 1.2 \\ 0.5 & -1.3 \end{bmatrix} \qquad C = \begin{bmatrix} 1.2 & 0.5 \\ 0.5 & 1.6 \end{bmatrix}$$

The magnitudes relevant for our issues of concern are the eigenvalues of M in the problem as just specified, i.e., with $\alpha = 1$, and for other values of α on the interval [0, 1]. In Table 1, the eigenvalues are reported for α equal to 1.0, 0.8, 0.6, 0.4, 0.2, and 0.0. They are reported in order of decreasing modulus for each α. The results for the actual problem at hand are given in the first column. It is readily seen to be one in which there is a unique stable solution, since there are two eigenvalues with moduli greater than 1.0. Thus −0.9365 and 0.4795 are the diagonal elements of the Λ_1 matrix if the latter is defined as relevant for seeking a determinate solution, i.e., as including the smallest (modulus) eigenvalues. But what is the MSV solution for the model? Since eigenvalues are continuous functions of the model parameters, it is clear that the second-listed eigenvalue in the first two columns is the "same" as the third eigenvalue in the remaining columns.[7] Thus the composition of Λ_1 relevant for the MSV solution includes 1.0887 and 0.4759. The MSV solution differs from the unique stable solution; indeed, the MSV

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[7] Here "same" is used in the following sense: for each specified eigenvalue, its value is a continuous function of each of the elements of the M matrix.

solution is dynamically explosive.

Reflection indicates that there is a simple way of generating examples of this type. Consider the m = 2 case, and suppose that the two rows of (1) represent separate univariate models. One of these can be specified so as to imply an explosive univariate solution (both moduli exceed 1.0) and the other to imply multiple stable solutions (both moduli less than 1.0). A pair of such models does not constitute a non-degenerate bivariate model, and will not permit RE solutions with some software.[8] But by simply adding very small non-zero values for one or more of the off-diagonal elements of A or C, a valid bivariate model of form (1) can be obtained. Yet with very small values for these off-diagonal elements, the eigenvalues for this bivariate model will be approximately the same as for the two univariate models taken together. Accordingly, there will be two stable and two explosive eigenvalues. The bivariate system will therefore be determinate; it will have one stable solution. The MSV solution for Ω must, however, involve one eigenvalue from each of the univariate models and will therefore differ from the unique stable solution.

An example of this type is provided by the univariate models defined by $a_{11} = -0.4$, $c_{11} = 1.5$ and $a_{22} = -1.5$, $c_{22} = 0.2$ with zeros elsewhere. The first has two explosive roots (–3.5549 and 1.0549) and the second has two stable roots (–0.8277 and 0.1611). To create a non-degenerate bivariate model we change a_{12}, a_{21}, c_{12}, and c_{21} from 0.0 to the values 0.01, 0.02, 0.02, and 0.01, respectively. Then the resulting eigenvalues for various values of α are as reported in Table 2. As in the example of Table 1, there is a unique stable solution for the model (i.e., with $\alpha = 1$) but it differs from the MSV solution.

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[8] A necessary rank condition is not satisfied. See, e.g., King and Watson (1998) or McCallum (1998).

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Table 1

Eigenvalues of M for various values of α

$\alpha = 1.0$	$\alpha = 0.8$	$\alpha = 0.6$	$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0.0$
−2.7022	−2.5402	−2.3615	−2.1593	−1.9211	−1.6156
1.0887	0.9267	−0.7961	−0.7108	−0.6066	−0.4585
−0.9365	−0.8702	0.7479	0.5456	0.3070	0.0000
0.4759	0.4096	0.3357	0.2505	0.1466	0.0000

Table 2

Eigenvalues of M for various values of α

$\alpha = 1.0$	$\alpha = 0.8$	$\alpha = 0.6$	$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0.0$
−3.5563	−3.3873	−3.2038	−3.0012	−2.7719	−2.5011
1.0551	0.8862	−0.7703	−0.7387	−0.7044	−0.6666
−0.8275	−0.7998	0.7027	0.5001	0.2707	0.0000
0.1610	0.1332	0.1038	0.0721	0.0378	0.0000

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