

E-stability when the exogenous AR(1) shock is not observable.

Consider the standard univariate set-up

$$\begin{aligned}\pi_t &= \mu + aE_t^* \pi_{t+1} + \gamma e_t \\ e_t &= \rho e_{t-1} + \varepsilon_t, \text{ for } |\rho| < 1,\end{aligned}$$

where e_t is an exogenous AR(1) process, which we usually treat as observable under learning. For the example discussed in McCallum (JME, 2009) we have $a = (1 + \mu_1)^{-1}$ with $\mu_1 > 0$, so that $0 < a < 1$, and $\gamma = -a$.

The usual MSV solution, which is also the unique nonexplosive solution, takes the form

$$\pi_t = k + b e_t,$$

where

$$k = (1 - a)^{-1} \mu \text{ and } b = (1 - \rho a)^{-1} \gamma.$$

The MSV solution can equivalently be rewritten as the AR(1) process $\pi_t = k(1 - \rho) + \rho \pi_{t-1} + b \varepsilon_t$, or

$$\pi_t = (1 - a)^{-1} \mu (1 - \rho) + \rho \pi_{t-1} + (1 - \rho a)^{-1} \gamma \varepsilon_t.$$

An AR(1) for π_t is clearly suitable for forecasting, so we consider the PLM (perceived law of motion)

$$\pi_t = c + d \pi_{t-1} + \eta_t,$$

where η_t is (unobserved) perceived white noise (equal to $b \varepsilon_t$ in the MSV solution). Under this PLM forecasts are given by

$$E_t^* \pi_{t+1} = c + d \pi_t,$$

assuming π_t is observable at t . The implied ALM (actual law of motion) is $\pi_t = \mu + a(c + d \pi_t) + \gamma e_t$, i.e.

$$\begin{aligned}(1 - ad)\pi_t &= (\mu + ac) + \gamma(1 - \rho L)^{-1} \varepsilon_t, \text{ or} \\ \pi_t &= \frac{(1 - \rho)(\mu + ac)}{1 - ad} + \rho \pi_{t-1} + \frac{\gamma}{1 - ad} \varepsilon_t.\end{aligned}$$

Thus the T-map from PLM to ALM parameters has components

$$T_c(c, d) = \frac{(1 - \rho)(\mu + ac)}{1 - ad} \text{ and } T_d(c, d) = \rho,$$

and the derivative matrix DT is triangular, with eigenvalues equal to the diagonal elements of DT , which are

$$\frac{(1-\rho)a}{1-ad} \text{ and } 0.$$

At the MSV solution $c = k(1-\rho)$, $d = \rho$ this first eigenvalue is $(1-\rho)a/(1-a\rho)$. Thus the E-stability condition for the MSV solution is

$$\frac{(1-\rho)a}{1-a\rho} < 1,$$

which holds since $0 < a < 1$ and $|\rho| < 1$. Thus the MSV solution is E-stable as usual for AR(1) PLMs for π_t .

We now consider E-stability of the (explosive) bubble solution discussed in McCallum, which for $\rho \neq 0$ is the process

$$\pi_t = -a^{-1}\mu + a^{-1}\pi_{t-1} - a^{-1}\gamma\rho^{-1}e_t.$$

This can be rewritten as the AR(2) process

$$\pi_t = -(1-\rho)a^{-1}\mu + (\rho + a^{-1})\pi_{t-1} - \rho a^{-1}\pi_{t-2} + \eta_t,$$

where $\eta_t = -a^{-1}\gamma\rho^{-1}\varepsilon_t$ is white noise. We can examine E-stability of this solution by looking at an AR(2) forecasting model for π_t , which nests the bubble solution (and also the MSV solution). Thus we consider the PLM

$$\pi_t = c + d\pi_{t-1} + f\pi_{t-2} + \eta_t.$$

Under this PLM, and assuming π_t is observable at t , forecasts are given by

$$E_t^* \pi_{t+1} = c + d\pi_t + f\pi_{t-1},$$

The ALM is $\pi_t = \mu + a(c + d\pi_t + f\pi_{t-1}) + \gamma e_t$, i.e.

$$(1-ad)\pi_t = (\mu + ac) + af\pi_{t-1} + \gamma(1-\rho L)^{-1}\varepsilon_t, \text{ or}$$

$$\pi_t = \frac{(\mu + ac)}{(1-ad)} + \frac{af}{(1-ad)}\pi_{t-1} + \frac{\gamma}{(1-ad)}(1-\rho L)^{-1}\varepsilon_t$$

which yields

$$\pi_t = \frac{(1-\rho)(\mu + ac)}{1-ad} + \left(\rho + \frac{af}{(1-ad)} \right) \pi_{t-1} - \frac{\rho af}{1-ad} \pi_{t-2} + \frac{\gamma}{1-ad} \varepsilon_t.$$

Thus the T-map has components

$$\begin{aligned} T_c(c, d, f) &= \frac{(1 - \rho)(\mu + ac)}{1 - ad}, \\ T_d(c, d, f) &= \rho + \frac{af}{(1 - ad)}, \\ T_f(c, d, f) &= \frac{-\rho af}{1 - ad}. \end{aligned}$$

The bubble solution

$$c = -a^{-1}\mu(1 - \rho), \quad d = (\rho + a^{-1}) \quad \text{and} \quad f = -\rho a^{-1}$$

is a fixed point of T , as is the MSV solution

$$c = (1 - a)^{-1}\mu(1 - \rho), \quad d = \rho, \quad f = 0.$$

The (d, f) components are autonomous, with derivative matrix

$$DT_{d,f} = \begin{pmatrix} \frac{a^2 f}{(1-ad)^2} & \frac{a}{(1-ad)} \\ \frac{\rho a^2 f}{(1-ad)^2} & \frac{-\rho a}{1-ad} \end{pmatrix}.$$

The eigenvalues of DT are the two roots of $DT_{d,f}$ and

$$DT_c = \frac{(1 - \rho)a}{1 - ad}.$$

For E-stability the three eigenvalues all need to have real parts less than one.

At the MSV solution the roots of $DT_{d,f}$ are 0 and $\frac{-\rho a}{1 - a\rho}$, which is less than one. In addition

$$DT_c = \frac{(1 - \rho)a}{1 - a\rho} < 1,$$

since $0 < a < 1$ and $|\rho| < 1$. Thus the MSV solution is E-stable, even when it is overparameterized as an AR(2) process.

At the bubble solution

$$\begin{aligned} DT_c &= 1 - \rho^{-1}, \quad \text{and} \\ DT_{d,f} &= \begin{pmatrix} -(a\rho)^{-1} & -\rho^{-1} \\ -a^{-1} & 1 \end{pmatrix}. \end{aligned}$$

From the DT_c condition, E-stability of the bubble solution requires $\rho > 0$. The E-stability conditions on $DT_{d,f}$ can equivalently be stated as requiring that the roots of $DT_{d,f} - I$ have negative real parts, which in turn is equivalent to $DT_{d,f} - I$ having a negative trace and positive determinant. But

$$\det(DT_{d,f} - I) = -(a\rho)^{-1},$$

and for $0 < a < 1$ and $\rho > 0$ we have $\det(DT_{d,f} - I) < 0$. It follows that the bubble solution cannot be E-stable.

The bubble solution discussed above does not exist if $\rho = 0$, but in this case there is another simple bubble,

$$\pi_t = -a^{-1}\mu + a^{-1}\pi_{t-1} + \eta_t,$$

where $\eta_t = a^{-1}\gamma e_{t-1}$ is white noise, which we are now treating as unobserved by agents when making their forecasts. For E-stability we can again consider the PLM $\pi_t = c + d\pi_{t-1} + \eta_t$. The E-stability calculation for the AR(1) PLM at the beginning of this note covers this case, with $\rho = 0$, and the E-stability condition for the intercept of the bubble solution fails since $\frac{(1-\rho)a}{1-ad} = \frac{(1-\rho)a}{0}$. Note that in this case the AR(1) coefficient of the ALM is $T_d(c, d) = 0$, and the bubble solution corresponds to a singularity (seen in the intercept), rather than a fixed point, of the T-map. The failure of E-stability for the bubble solution in the $\rho = 0$ case is discussed further in Appendix A of Evans (JME, 1989).

Conclusion: Treating the exogenous AR(1) process e_t as unobserved, both the MSV solution and the bubble solution can be represented as AR processes. The MSV solution is stable under learning and the bubble solution is unstable under learning.

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