A mechanism design approach to a common distributional problem of centrally administered organizations*

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Abstract

We study the problem of the planner of a centrally administered organization who is implicitly bound by a budget constraint, and needs to make purchasing decisions for a set of exhaustible indivisible goods based on the information collected from different comprising units. While a concrete motivation for our work comes from the resource distribution problems faced by the central logistic headquarters of the Turkish Navy, several other organizations such as universities, federal institutions, and nationwide companies are also subject to similar problems. Our approach allows the planner to first consider a feasible *hypothetical set of objects* through which she determines the *actual set of objects* to be purchased based on the (preference) information collected from each unit. This approach can also be seen as a new way to recover compatibility between efficiency and equity. For each of two different cases of this problem we identify an efficient, equitable [in the sense of equal-treatment-of-equals, or envy-freeness], and strategy-proof mechanism that Pareto dominates all other equitable and strategy-proof mechanisms. This mechanism can be interpreted as a combined analogue of a Vickrey-Groves mechanism and that of the student-optimal stable mechanism of two-sided matching.

JEL Classification: *C71; C78; D71; D78*

1 Introduction

The standard approaches to a typical indivisible good allocation problem (as well as those to a general resource allocation problem) assume the presence of a central planner (or, mechanism designer) who has complete information about the resources at hand.[1] This assumption is justified

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[1] There is clearly a large body of literature dedicated to the study of informational incompleteness/imperfections and asymmetries from the participants' and that of the central planner's point of view. These are not the subject of our paper.

when resources are fixed, non-exhaustible, and are repeatedly reallocated over time (e.g., allocation of houses to applicants, offices to staff, or tasks to workers etc.) It is not, however, otherwise (e.g., allocation of supplies, food, clothing, equipment, etc.), and the central planner has to make purchasing decisions about what and how much to buy prior to the allocation decision. Consider, for example, the director of a department who needs to appoint new faculty to the department based on the recommendations of an internal recruiting committee. Such a director's hiring decision is bound by at least two considerations: The budget allocated for the expansion of the department, which specifies *maximum* spending on new faculty; and the particular demands (preferences) of the department outlined by the recruting committee. For instance, it may be that the department's budget permits hiring for at most two junior positions while the recruiting committee recommends one offer to a senior member and one to a junior one. In this case, a resolution could be making one junior offer, and letting one position go unfilled for that year. In such a situation, excess funds would be turned over to the following year's hiring budget, or may be spent for other needs of the department.[2]

Our goal in this paper is to study a critical distributional problem common to several centrally administered organizations. A concrete motivation for our work is derived from the problems encountered in the Turkish Navy during the provision of logistic support for the comprising naval units. This application will constitute our running example throughout the paper.[3] The distribution problem of the Turkish Navy can roughly be described as follows:[4] The naval force consists of several naval bases each located at a particular location. Each base is divided into a number of naval units. The allocation of logistic resources among all units takes place via a centralized procedure as follows:[5] Around the middle of each year, each unit submits a survey to the Turkish Naval Headquarters listing an estimate of their needs for the following year in a certain order. This order, which can be interpreted as a preference order, for example, may be determined according to the urgency or priority of the particular item listed. For each category of needs, each unit submits a list of their needs. These categories typically include equipment, electronic devices, consumable supplies, clothing, petroleum, oils, lubricants, medical supplies, office supplies, weapons, and so on. For each category of goods, upon the receipt of the lists from all the units, the naval headquarters carry out the allocation process, in which each unit receives a subset of the items it has listed, in two steps:

(1) An initial assessment with regards to how much of each good is needed to purchase from outside sources based on the available budget,

(2) An allocation of the purchased goods among units.

Any unused excess funds are returned to the treasury for later use, or used for other purposes. The Turkish Navy sets fourteen major goals for the Logistic Support Headquarters. Among these are the following five:

[2] This example could be made more interesting by introducing several departments competing for the limited slots the school has, where each department has its own list of candidates to extend a possible offer to.

[3] We strongly suspect that similar problems arise in various different areas as well.

[4] For classification reasons we are unable to disclose any actual figures.

[5] In the Turkish Navy, some allocation decisions are made by the Central Turkish Naval Headquarters for the allocation of goods across all naval bases. There are also similar central decision systems/procedures made within each naval base.

- *"In accordance with the principles outlined by the Turkish Naval Force, to constantly and economically provide and maintain logistic support to all major and minor naval units in the best possible way, and hence contribute to the readiness of all units to the utmost level."*

- *"To function with and help establish perfect harmony and coordination within all the naval units of the Turkish Naval Force."*

- *"To minimize the possibility of accumulation of any unused resources."*

- *"To ensure resource saving by increasing efficiency."*

- *"To decrease the variety of systems, equipment, and weapons due to the high purchasing and storage costs caused by such variety."*

The above stated goals of the Turkish Navy suggest that such centrally administered organizations highly value efficient, equitable, and balanced (homogenous) distribution of resources among all units. These are probably most natural concerns for a military organization where units are in close contact and constantly interact with one another. To give a specific example, consider a unit that gives high priority in its survey to a special encryption coding & decoding device for secretly communicating with other units on emergent matters. Then the central headquarters certainly cannot consider this particular unit independent of the other units it is connected with, which might also need a similar system installed.

There are other organizations facing similar situations with similar concerns. Consider the following simple example: Imagine a dean who decides to re-model the university with modern furniture. Suppose that she brings the list of possible kinds of furniture to be bought down to five options, and would also like to give the opportunity to the director of each department to suggest his/her own preferences among the five (which we can assume to be a strict ranking of the five options). It is natural to think that the dean is not willing to allow for a different kind of furniture in each department, since one can naturally expect the buildings of the university to share certain common features. Suppose further that the dean, in order to satisfy as many department directors as possible, allows for two selected options within the university. That is, once the two possibilities are determined (based on the collection of suggestions), each department will be free to use its favorite kind of furniture between the two. This would in a way also guarantee a balanced distribution of furniture types within the university while allowing for certain degree of variation across different buildings as well.

Given the concern for efficiency and equity/balancedness, a unique feature of the decision problem of the naval headquarters is that steps (1) and (2) are not independent of each other, and hence cannot be considered in isolation. Put differently, even though it is true that available resources determine the allocation to be made, an efficient and equitable/balanced allocation simply may not exist for any given set of resources.

To attain equitable and balanced allocation of resources, we restrict attention to mechanisms that achieve certain equity criteria *ex post*. One such criterion that has been often studied in the literature is *no-envy* (Foley [12]): Suppose an allocation is made in which a need of each unit is fulfilled. No unit ranks in its list a need of some other unit which is fulfilled as a higher priority need than its own fulfilled need. While the no-envy criterion is imposed and studied in a number of economic contexts as a critical justice and fairness criterion, in our situation it serves two other important purposes as well: achieving an efficient allocation and helping obtain a balanced (variety-saving) dispersion of the fulfilled needs among units.

Equitable distribution of resources and efficiency are two important principles for almost all economic applications. One, however, often faces a trade-off between the two, and the allocation of indivisible goods is one where this is felt most strongly. The literature so far considered two ways to recover compatibility between the two requirements: (1) introduction of side-payments (see, for example, Vickrey [29], Groves [14], Alkan, Demange and Gale [5], Tadenuma and Thomson [27], Klijn [17], Abdulkadiroglu, Sönmez and Unver [3]); (2) allowing for randomization (see, for example, Hylland and Zeckhauser [15], Zhou [30], Abdulkadiroglu and Sönmez [1], and Bogomolnaia and Moulin [5]). For the particular problem we are dealing with, side payments are clearly not an option. On the other hand, the use of lotteries would make it possible to achieve *ex ante equity.* However, it is quite possible that the resulting allocations may fail to achieve a balanced distribution of resources. This, in particular, is the case for one of the most common real-life allocation mechanisms, the random priority[6] (a.k.a., the random serial dictatorship).

One intriguing aspect of the central planner's problem now is that the purchasing decision has to be made after the preferences of each unit have been taken into account. Otherwise, if goods are initially purchased regardless of preferences, then some of the goods could end up being unallocated, and thus resulting in inefficiency. Unallocated goods could also be stored for future use, however this is usually not preferred because of storage costs (for example, the Turkish Naval Headquarters strongly discourages such a practice through the third goal stated above) and the unnecessary limitation imposed on other potential uses of the financial budget.

We offer the following intuitive first approach to this problem: The central planner initially considers a *hypothethical set of objects* which is affordable given the budget constraint. (Clearly, such a set may not be unique.) Next, he uses an allocation mechanism that produces a 'desirable' distribution of the objects in this set, which presumably, may not assign every object in the hypothetical set to someone, and he finally obtains the *actual set of objects* to be purchased by omitting all unassigned objects from the hypothetical set. This approach, in this sense, can be seen as a third way to recover compatibility between efficiency and equity.

We consider two interesting cases of this problem: (1) The number of units of each object (in the hypothetical set) is exactly one; (2) The number of units of an object (in the hypothetical set) may be different than one. For the former case, we identify an efficient (with respect to the actual set of resources), and strategy-proof mechanism that treats equals equally. Most notably, it Pareto dominates any other strategy-proof mechanism that treats equals equally (Proposition 2). It is also envy-free (Corollary 1). For the latter case, we again identify an efficient (with respect to the actual set of resources), strategy-proof, and envy-free mechanism. Furthermore, it Pareto dominates any other strategy-proof and envy-free mechanism (Theorem 1). In order to determine if a unit can be assigned a particular object, these mechanisms first subject such an object to an "attainability" test based on the preferences of all units but the particular unit itself, and assign each unit its best attainable object. Interestingly, the attainability test of an object for a particular unit turns out to be a recursive procedure. The proposed mechanisms also satisfy

[6] Random priority works as follows: Consider a collection of distinct objects. Choose a random ordering of agents (from a uniform distribution), let the first agent choose his favorite object, next let the second agent choose his favorite object from whatever remains, and so on. Clearly, the outcome of such a procedure could be quite disappointing for those with a late turn in the ordering. Use of random priority in our setup is also likely to yield an unbalanced distribution.

a certain monotonicity property with respect to the hypothetical set of objects (Remark 3). This property proves quite useful in bringing down the number of possible hypothetical sets that the central planner can take into consideration.

From a broader angle, for two reasons the proposed mechanisms can be interpreted to be the analogoues of the celebrated Vickrey-Groves mechanisms for the present context. First, both kinds of mechanisms determine a participant's assignment based *explicitly* on all participants' preferences but those of the particular participant. Second, Vickrey-Groves mechanisms achieve strategy-proofness and efficiency at the cost of budget unbalancedness. That is, an outside intervention may be required to introduce additional compensation levels, or to collect excess budget surpluses to implement the outcome. In our case, similarly, there may be excess funds (or excess hypothetical objects that can be transformed into currency at zero cost) that need to be returned to the funding source allowing their use for other purposes and thereby maintaining effficiency. Differently from general Vickrey-Groves mechanisms, our mechanisms are always envy-free. On another account, the proposed mechanisms can also be seen to be the analogues of the well-known student-optimal stable mechanism of two-sided matching theory, which is also the unique stable (and also strategy-proof) mechanism that Pareto dominates any other stable mechanism in its context.

2 The model

Let $N \equiv \{1, 2, \ldots, n\}$, $n \geq 2$, denote the finite set of units (e.g., military, academic, corporative, etc.) that comprise an organization. The central planner of the organization has a maximum of $r \in \mathbb{R}_{++}$ to spend on the total needs of all units. We call any possible item that a unit might list in its preference (necessity) list as an **object** (e.g., a particular equipment, technological upgrade, internal necessity, new facility etc.) In this paper, we distinguish between the set of objects the central planner can choose to allocate from if she wishes, and the set of objects that are eventually allocated to all units. Ours, to the best of our knowledge, is the first paper that models an indivisible good allocation problem in this fashion.

Given the current market prices of each object that a unit might list in its needs list, let X_H denote the (finite) **hypothetical set of objects.** Let $\mathcal{X}$ denote the collection of all possible hypothetical sets of objects that the central planner can afford to purchase given her maximum spending r. Let the number of copies of each object available in the hypothetical set be one. Also available to each unit is an outside option, called the **null object**. Let 0 denote this object. Let supply of the null object be n, i.e, the null object can be assigned to any number of units. Let $\widetilde{X}_H = X_H \cup \{0\}$. Each unit $i \in N$ is equipped with a *complete*, *transitive,* and *anti-symmetric*[7] relation R_i on $\widetilde{X}_H$. Let $\mathcal{R}$ denote the class of all such preferences. Let P_i be the strict relation associated with R_i. Let $R = (R_i)_{i \in N}$ be a preference profile. Then a **problem** is a pair $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$.

For a given problem (X_H, R), a feasible **allocation** for X_H is a list $\alpha \equiv (\alpha_i)_{i \in N}$ such that for each $i \in N$, $\alpha_i \in \widetilde{X}_H$, no unit is assigned more than one object and no object in X_H is assigned to more units than its supply (which is one in this case). Let $\mathcal{A}(X_H)$ be the set of all feasible

[7] A preference relation R_i on $\widetilde{X}_H$ is *anti-symmetric* if for each $x, y \in \widetilde{X}_H$, $x \; R_i \; y$ and $y \; R_i \; x$ implies $x = y$.

allocations for X_H. An allocation $\alpha \in \mathcal{A}(X_H)$ induces an **actual set of objects** X_α which is the set of objects assigned at α. Formally, X_α is represented by a pair $(Y_\alpha, s_{\alpha,0})$ where $Y_\alpha \equiv \cup_{i \in N} \alpha_i$ and $s_{\alpha,0} = |\{i \in N : \alpha_i = 0\}|$ is the supply of the null object. Previous models assume that the hypothetical set of objects is in fact the actual set of objects, which already exists prior to the decision of the central planner.

An allocation is **(Pareto) efficient** (with respect to the actual set of objects it induces), if it is not possible to make a unit better off without making another worse off by a reallocation of objects within the induced actual set of objects, i.e., given (X_H, R), the allocation $\alpha \equiv (\alpha_i)_{i \in N}$ feasible for X_H that induces the actual set of objects X_α is **efficient** if there is no α' feasible for X_α such that $\alpha'_i \; R_i \; \alpha_i$ for each $i \in N$ and $\alpha'_j \; P_j \; \alpha_j$ for some $j \in N$. It is also possible to interpret efficiency as a *reallocation-proofness* requirement. Unlike most others, in our model efficiency will be an auxiliary requirement, which alone does not have much bite. (Note that assigning each agent the null object is also efficient.)

Given $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$, an allocation $\alpha \in \mathcal{A}(X_H)$ is **Pareto dominated** by $\beta \in \mathcal{A}(X_H)$, $\beta \neq \alpha$ if $\beta_i \; R_i \; \alpha_i$ for each $i \in N$, and $\beta_i \; P_i \; \alpha_i$ for some $i \in N$.

A **mechanism** is a function that associates to each problem (X_H, R) an allocation α feasible for X_H. Let φ denote a generic mechanism.

Given a problem $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$, and a unit $i \in N$, let $\varphi_i(X_H, R)$ denote unit i's assignment at (X_H, R). Let R_{-i} denote the profile $R_{N \setminus \{i\}}$. We next introduce some of the basic properties of mechanisms.

Efficiency: For each $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$, the allocation $\varphi(X_H, R)$ is efficient.

A mechanism φ is **Pareto dominated** by another mechanism ϕ if for each $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$, each $i \in N$, $\phi_i(X_H, R) \; R_i \; \varphi_i(X_H, R)$ where the relation is strict for some $i \in N$ and $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$.

Next are two central properties studied in this paper. The first one can be seen as a minimal equality requirement: The mechanism should not discriminate between any two units who have the same preferences.

Equal Treatment of Equals: For each $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$, and each $i, j \in N$, if $R_i = R_j$, then $\varphi_i(X_H, R) = \varphi_j(X_H, R)$.

Our second property has been extensively studied in many contexts. It requires that no unit ever gains by misreporting its preferences.

Strategy-proofness: For each $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $\varphi_i(X_H, R) \; R_i \; \varphi_i(X_H, (R'_i, R_{-i}))$.

The following property is often imposed in the literature: If a unit's assignment does not change when its preferences change, nobody else's does either.

Nonbossiness: For each $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$, each $i \in N$, and each $R'_i \in \mathcal{R}$, if $\varphi_i(X_H, R) = \varphi_i(X_H, (R'_i, R_{-i}))$, then $\varphi(X_H, R) = \varphi(X_H, (R'_i, R_{-i}))$.

3 A New Mechanism

We start with a negative result.[8]

The null mechanism (Z): For each $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$ and each $i \in N$, $Z_i(X_H, R) = 0$.

Proposition 1 The null mechanism is the only mechanism that satisfies equal treatment of equals, strategy-proofness, and nonbossiness.

Proposition 1 implies that no interesting mechanism satisfies *equal treatment of equals, strategy-proofness*, *nonbossiness*. Nonbossiness is usually referred to as a mild condition although many times its desirability is questionable. Therefore, we give up *nonbossiness*, and turn our attention to *strategy-proof* mechanisms that satisfy *equal treatment of equals*. We introduce the following mechanism: Assign each unit its most preferred object among the ones that no remaining unit prefers to the null object. In other words, if two units both prefer some object a to the null object, then neither of them gets it.

We need some extra notation before we formally define this new mechanism. Given $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$ and $X' \subseteq \widetilde{X}_H$, let $f(R_i, X') \equiv \{a \in X' : a \; R_i \; x \text{ for all } x \in X'\}$. That is $f(R_i, X')$ is the favorite object of unit i in X'. Given $i \in N$, $a \in \widetilde{X}_H$, let $U_a(R_i) \equiv \{x \in \widetilde{X}_H : x \; P_i \; a\}$ and $L_a(R_i) \equiv \{x \in \widetilde{X}_H : a \; P_i \; x\}$. That is, $U_a(R_i)$ is the set of objects unit i prefers to a whereas $L_a(R_i)$ is the set of objects to which i prefers a.

Mechanism φ^*: For each $R \in \mathcal{R}^N$ and each $i \in N$, $\varphi^*_i(X_H, R) = f(R_i, \widetilde{X}_H \setminus \cup_{j \neq i} U_0(R_j))$.

Note that this mechanism is well-defined. As the next result suggests, it is the most appealing one among *strategy-proof* mechanisms that satisfy *equal treatment of equals*.

Proposition 2 Mechanism φ^* is efficient, strategy-proof and satisfies equal treatment of equals. If any mechanism other than φ^* and Z also satisfies strategy-proofness and equal treatment of equals, then it is Pareto dominated by mechanism φ^*, and it Pareto dominates Z.

Proposition 2 shows that from a welfare standpoint, mechanism φ^* is the best and the null mechanism is the worst mechanism within the class of mechanisms satisfying *strategy-proofness* and *equal treatment of equals*. Note also that Propositions 1 and 2 still hold if one replaces *equal treatment of equals* by *anonymity*[9] (the mechanism does not depend on the naming of units) which is a stronger *equality* requirement. Furthermore, mechanism φ^* and the null mechanism are not

[8] Another similar negative result from the literature that assumes the actual set is given a priori is due to Svensson [26]: The only *strategy-proof*, *neutral*, and *nonbossy* mechanisms are serial dictatorships.

[9] Formally, φ satisfies *anonymity* if given any permutation $\pi : N \to N$ of agents, for each $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$ and each $i \in N$, we have $\varphi_i(X_H, R^\pi) = \varphi_{\pi(i)}(X_H, R)$ where $R^\pi \equiv (R_{\pi(i)})_{i \in N}$.

the only mechanisms satisfying *strategy-proofness* and *anonymity*. The following one is another example: For each $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$ and each $i \in N$, if for each $\{j,k\} \in N\backslash\{i\}$, $R_j = R_k$, then $\varphi_i(X_H, R) = f(R_i, \widetilde{X}_H \backslash \cup_{j \neq i} U_0(R_j))$, otherwise $\varphi_i(X_H, R) = 0$.[10]

In fact, *strategy-proofness* together with *equal treatment of equals* implies a much stronger form of equality which is also the central property in the next section: No unit ever prefers some other unit's assignment to its own.

No-Envy (Foley [12])**:** For each $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$ and each $i, j \in N$, we have $\varphi_i(X_H, R)\ R_i\ \varphi_j(X_H, R)$.

No-envy is also an appealing requirement studied in various models such as classical exchange economies, the extension of our model to the case where monetary compensations are possible, the division of a heterogenous good, the allocation of an infinitely divisible commodity to a set of units with single-peaked preferences etc. (see Thomson [28] for other applications of this concept). If a mechanism satisfies *no-envy*, it is said to be *envy-free*. The following lemma is an immediate consequence of Proposition 2 and Claim 1 in the proof of Proposition 2.

Lemma 1 *If a mechanism satisfies strategy-proofness and equal treatment of equals, then it is envy-free.*

Fleurbaey and Maniquet [11] show that under certain richness assumptions the no-envy correspondence (which selects all the envy-free allocations) stands out as the only solution concept to satisfy the well-known implementability condition *Maskin monotonicity* (which is implied by *strategy-proofness*), *equal treatment of equals*, and *nonbossiness*. Clearly, Lemma 1 is independent of that result (also recall Proposition 1).[11]

Corollary 1 *Mechanism* φ^* *is envy-free.*

Corollary 2 *Mechanism* φ^* *Pareto dominates any other envy-free mechanism that satisfies strategy-proofness.*

A special case of this model is when there is a single object to be assigned, i.e., $|X_H| = 1$. The only *strategy-proof* and *nonbossy* mechanisms are *hierarchical mechanisms:* There is a group of units and a priority order within this group such that the mechanism assigns the object to the unit with the highest priority among those who prefer getting the object to not getting it (Pápai [19]). By Proposition 1, these mechanisms do not meet *equal treatment of equals*. Mechanism φ^* when applied to this context simplifies to the following: If there is only one unit who prefers getting the object to not getting it, then the unit is assigned the object, otherwise no unit is assigned the object.

Corollary 3 *Suppose* $|X_H| = 1$. *The mechanism that assigns the object only if there is exactly one unit who prefers getting it to not getting it Pareto dominates any other mechanism satisfying strategy-proofness and equal treatment of equals.*[12]

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[10] Clearly, given a problem (X_H, R) if for each $j,k \in N$, $R_j = R_k$, then for each $i \in N$, $\varphi_i(X_H, R) = 0$. We leave it to the reader to check that this rule satisfies the two properties.

[11] When monetary compensations are allowed, Sakai [23] shows that the *no-envy* correspondence is the only solution satisfying *Maskin monotonicity*, *equal treatment of equals*, and a *neutrality* condition.

[12] If *equal treatment of equals* is replaced by *anonymity*, we obtain uniqueness.

4 An Extension: Multiple Supplies

An important extension of the problem we previously considered is when there may be multiple copies of a particular object. This extension too attracted much attention in the recent literature. A popular application is the *school choice problem* (Abdulkadiroglu and Sönmez [1]): There is a set of schools (objects) each of which has a certain number of seats and a set of students each of whom seeks one seat at one of the schools.[13] Most of the papers concerning this extension assume that an exogenous *priority order* over units is given for each object. If there is a shortage of seats at a particular school that is overdemanded, then the priority order for that school can be used to determine the allocation (more on this after Theorem 1). In contrast with these papers, we do not make a priori distinction among units.

First we provide the formal extension. With an abuse of notation let (X_H, s_H) denote the (finite) **hypothetical set of objects** such that X_H is the set of object types available in the hypothetical set of objects and for each $x \in X_H$, $s_{H,x} \in \mathbb{Z}_{++}$ is the supply of x. Let $s_H \equiv (s_{H,x})_{x \in X_H}$ be the supply vector. Let $\widetilde{X}_H \equiv X_H \cup \{0\}$. Let supply of the null object be n. Each unit $i \in N$ is equipped with a *complete*, *transitive* and *anti-symmetric* preference relation R_i over $\widetilde{X}_H$. Let $\mathcal{R}$ denote the class of all such preferences. Let P_i denote the strict relation associated with R_i. Let $R = (R_i)_{i \in N}$ be a preference profile. A **problem** is a triple (X_H, R, s_H).

For a given problem (X_H, R, s_H), a feasible **allocation** for (X_H, s_H) is a list $(\alpha_i)_{i \in N}$ such that for each $i \in N$, $\alpha_i \in \widetilde{X}_H$, no unit is assigned more than one object, and no object $x \in X_H$ is assigned to more units than its supply. Let $\mathcal{A}(X_H, s_H)$ be the set of all feasible allocations for (X_H,s_H). A given allocation $\alpha \in \mathcal{A}(X_H, s_H)$ induces an **actual set of objects** X_α which is the set of objects assigned at α. Formally, X_α is denoted by a pair (Y_α, s_α) where $Y_\alpha \equiv \cup_{i \in N} \alpha_i$, $s_\alpha \equiv (s_{\alpha,x})_{x \in Y_\alpha}$ and $s_{\alpha,x} = |\{i \in N : \alpha_i = x\}|$ for each $x \in Y_\alpha$.

A **mechanism** φ is a function that chooses an allocation for each problem (X_H, R, s_H). Given (X_H, R, s_H), a **subproblem** is a triple $(X_H, R_{-N'}, s_H - |N'| * \mathbf{1})$ for some $N' \subset N$, $N' \neq \varnothing$, where $\mathbf{1}$ is the $|X_H| \times 1$ vector of 1's, obtained from (X_H, R, s_H) by removing all units in N' and reducing the supply of each object by $|N'|$. Note that *an object may have a negative supply in a subproblem.* Let $\mathcal{P}_{(X_H,R,s_H)}$ be the set consisting of all subproblems of (X_H, R, s_H) and itself, i.e, $\mathcal{P}_{(X_H,R,s_H)} \equiv \underset{N' \subset N}{\cup} (X_H, R_{-N'}, s_H - |N'| * \mathbf{1})$.

4.1 Optimality, equity, and balancedness

We first focus on three important criteria that would possibly be expected to be satisfied by an allocation decided upon by a centrally administered organization that serves several units: *optimality, equity*, and *balancedness* (i.e., homogeneity in the resource distribution). The central property studied in this paper is no-envy. No-envy allows the central planner to achieve ex post equality among units. But also, as we will shortly argue it has close ties with the notions of efficiency and balancedness.

Probably not much suprisingly, the aforementioned three requirements are in conflict. We start with an informal discussion of the tension and the relationship between them. First is the tension between optimality and balancedness. Even though we will not formally define and study

[13] See for example Ergin [10], Abdulkadiroğlu and Sönmez [2], and Kesten [16]. This application is also closely related to the college admissions problem (Gale and Shapley [13]).

"balancedness" and instead, point out its close connection with no-envy, we use this term in the sense of allocating as few different object types as possible. Indeed, as made clear by the fifth stated goal of the Turkish Logistic Headquarters, allocations that introduce less variety of object types are deemed more desirable.

Suppose there are five units, and each unit submits the below given preferences to the central planner. Suppose that the central planner can afford to buy (i.e., her hypothetical set) five copies for each object. Clearly, now the 'optimal' allocation is the one where each unit receives its favorite object (denoted below by dots). However, this allocation overall offers two types of objects, and can easily be argued to be less 'balanced' than the allocation which offers each unit object a (denoted below by underscores).

R_1	R_2	R_3	R_4	R_5
$\cdot\underline{a}$	$\cdot\underline{a}$	$\cdot\underline{a}$	$\cdot\underline{a}$	$\cdot b$
b	b	b	b	$\underline{a}$

Next consider the following preference profile where the hypothetical set of the central planner contains three copies from each of objects a, b, and c. The allocation illustrated with boxes below gives each unit its favorite object, and overall offers three types of objects. However, it is less 'balanced' than the allocation illustrated with the underscores, which overall offers only two types of objects. On the other hand, the allocation with the underscores is Pareto inferior to the one with boxes.

R_1	R_2	R_3	R_4	R_5
$\boxed{\cdot\underline{a}}$	$\boxed{b}$	$\boxed{b}$	$\boxed{\cdot c}$	$\boxed{b}$
b	$\underline{\cdot c}$	$\cdot\underline{a}$	$\underline{a}$	$\underline{\cdot c}$
c	a	c	b	a

Interestingly, if balancedness has critical importance for the central planner, then for this example she can still do better than the allocation with the underscores. Observe first that the allocation with the underscores is not envy-free (since unit 4 is envious of units 2 and 5 for object c). It is indeed possible to make a Pareto improvement on this allocation, while preserving its balancedness (in the sense of not introducing any new objects for allocation). This can simply be done by assigning the envious unit (in this case, unit 4) the object it is envious for (in this case, object c). This small update in the allocation with the underscores gives us the allocation above with the dots. Now the allocation with the dots is envy-free. Most notably, the Pareto improvement over the allocation causing envy does not disrupt the balancedness of the former allocation (i.e., the allocation with the underscores). The reason for this is simple: Since the new assignment of the (previously) envious unit is an object that was already assigned to some unit, the new allocation can not be introducing types of objects that were previously unassigned.

In fact, the above kind of updating can similarly be done on any allocation causing envy at which there are several envious units again in a way that leads to an envy-free allocation that is Pareto superior to and no less balanced than the initial allocation. The following procedure makes this point precise:

Take any allocation causing envy where there are multiple envious units. Consider only those envious units each of whom most envies a non-envious unit (i.e., the nonenvious unit is currently assigned its favorite object among the assigned ones). Update each such unit's assignment as *described above until each remaining envious unit is most envious only of another envious unit. This implies that at this point there is at least one envy-cycle among envious units (i.e., unit i_1 is envious of some unit i_2; unit i_2 is envious of some unit i_3; ...; unit i_k is envious of unit i_1). Within each envy-cycle carry out the trades that are in the best interest of each unit in that cycle. Consider the new allocation, and go to the very beginning of this procedure. Repeat the procedure until the final allocation is envy-free.*

Indeed, this observation gives us a second reason for imposing the no-envy requirement: *Any allocation at which there are envious units can be improved upon without any harm to its degree of balancedness.*

Our third reason for searching within the class of envy-free mechanisms is that no-envy also guarantees efficiency (or, reallocation-proofness). Interestingly, this has been a well-known and often used implication for indivisible good allocation problems when monetary compensations are also allowed (see Svensson [25] and Alkan, Demange and Gale [5]).

Lemma 2 *Let φ be an envy-free rule. Then it is also efficient.*

4.2 Two new mechanisms

In this larger setting, we continue to study our strategic and equity properties. Our central ex-post equality requirement is *no-envy*. We first introduce a new mechanism, the *unrestricted fair (UF) mechanism.* The outcome of this mechanism is calculated via repeated applications of the UF *procedure*. In this procedure, each unit initially *demands* its favorite object among the available ones, if the competition for a particular object is overwhelming, that is, if the number of demands for the object exceeds its supply, then such an object is discarded from the set of available objects. At the next application, the UF procedure is applied to the set of remaining objects, and the procedure is repeated until no object is discarded any more. Clearly, the procedure terminates in a finite number of applications.

UF procedure: *Let a problem or a subproblem be given. Each unit demands its favorite object among the available ones. If for any object the number of units who demand it is greater than its supply, then remove it from the set of available objects. (If an object's supply is negative, it is simply removed from the set of available objects since the number of units who demand it is always non-negative.)*

Unrestricted fair mechanism UF: *Let a problem (X_H, R, s_H) be given. Repeatedly apply the UF procedure until there is no object with greater demand than supply. The unrestricted fair mechanism assigns each unit what it demands at the last application of the procedure in which there is no object with greater demand than supply.*

We will simply say that *an object is eliminated by UF* in a problem whenever it is eliminated from the set of available objects at some application of the UF procedure to that problem. As established in the following proposition the unrestricted fair mechanism is envy-free. Furthermore, it is Pareto superior to any other mechanism that is also envy-free.

Proposition 3 *The unrestricted fair mechanism is envy-free. Furthermore, it Pareto dominates any other envy-free mechanism.*

Corollary 4 *The unrestricted fair mechanism is efficient.*

It is easy to see that the unrestricted fair mechanism is not strategy-proof. In the remainder of the paper it is our objective to recover strategy-proofness.

Definition 1: *Let a problem (X_H, R, s_H) be given.* ***Object* a *is scarce for unit* i *in*** *(X_H, R, s_H) iff for each $x \in U_a(R_i) \cup \{a\}$*
either (i) x is eliminated by UF in the subproblem $(X_H, R_{-i}, s_H - \mathbf{1})$.
or (ii) x is scarce for some unit j in the subproblem $(X_H, R_{-i}, s_H - \mathbf{1})$.

Definition 1 provides a notion of scarcity which could be interpreted as an 'attainability' test. Given a problem (X_H, R, s_H), take a unit $i \in N$ and an object $a \in X_H$. In order to determine if object a is scarce for unit i in (X_H, R, s_H), one needs to consider each object in the upper contour set of R_i at a as well as a itself. For each such object we first check if the object is eliminated by the UF procedure in the subproblem obtained from (X_H, R, s_H) by removing unit i and reducing the supply of each object by 1. If not, we check if there is a unit for whom the object is scarce in the subproblem mentioned above. For object a and every object in the upper contour set of R_i at a, if either one of the two parts of the scarcity definition holds, then (and only then) we conclude that a is scarce for unit i in (X_H, R, s_H). Note that the *null object is never scarce for any unit in any (sub)problem.*

The scarcity definition is recursive, and makes extensive use of the UF procedure. (Also see the Appendix for an example that illustrates this definition.) It is also worthwhile to observe that if one ever concludes that an object a is scarce for a unit i in a (sub)problem, then at some point, (i) of the scarcity definition should hold for a so that the test of the definition terminates. In other words, when checking for the scarcity of a for unit i in a (sub)problem I, we should always end up in a subproblem of I, say I', in which a is eliminated by UF. Let $S(i, I)$ denote the set of objects that are scarce for i in (sub)problem I.

Remark 1 *Let a problem $(X_H, R, s_H), I \in \mathcal{P}_{(X_H,R,s_H)}$, $a \in X_H$ and $i \in N$ be given. Then $a \in S(i, I)$ iff $(U_a(R_i) \cup \{a\}) \subseteq S(i, I)$.*

Remark 2 *Let a problem (X_H, R, s_H), $I \in \mathcal{P}_{(X_H,R,s_H)}, a \in X_H$ and $i \in N$ be given. Let $a \in S(i, I)$. Then for each $x \in U_a(R_i) \cup \{a\}$, there is a subproblem I' of I, such that i is not in the unit set of I' and x is eliminated by UF in I'.*

We are now ready to present our main solution to the problem considered in this study. The scarcity notion defined above is an essential part of this mechanism. In each problem, this mechanism assigns each unit its favorite object among the ones that are not scarce for it in that problem.

Mechanism ψ^*: *For each problem (X_H, R, s_H) and each unit $i \in N$, ψ^* assigns unit i its most favorite object among the ones that are not scarce for i in (X_H, R, s_H).*

The next result shows that mechanism ψ^* is indeed well defined. It is followed by our main result.

Lemma 3 *For each problem* (X_H, R, s_H), ψ^* *chooses an allocation feasible for* (X_H, s_H).

Theorem 1 *Mechanism* ψ^* *is strategy-proof and envy-free. Furthermore, it Pareto dominates any other envy free and strategy-proof mechanism.*

Corollary 5 *Mechanism* ψ^* *is efficient.*

Corollary 6 $\psi^*(X_H, R, s_H) = \varphi^*(X_H, R)$ *for each* (X_H, R, s_H) *such that* $s_H = \mathbf{1}$.

The equivalence between the two mechanisms in the single supply case follows from the following observation: $U_{\varphi_i^*(X_H,R)}(R_i) \subseteq S(i, (X_H, R, s_H)) \subseteq \cup_{j\neq i} U_0(R_j)$ for each $i \in N$ and each (X_H, R, s_H) such that $s_H = \mathbf{1}$. To see this, let $i \in N$ and (X_H, R, s_H) be such that $s_H = \mathbf{1}$. For the first relation, note that by definition of φ^*, $U_{\varphi_i^*(X_H,R)}(R_i) \subseteq \cup_{j\neq i} U_0(R_j)$. Note also that each $x \in \cup_{j\neq i} U_0(R_j)$ is eliminated by UF in $(X_H, R_{-i}, s_H - \mathbf{1})$ because each has a zero supply. The previous two statements imply that each $x \in U_{\varphi_i^*(X_H,R)}(R_i)$ is eliminated by UF in $(X_H, R_{-i}, s_H - \mathbf{1})$, which in turn implies that $U_{\varphi_i^*(X_H,R)}(R_i) \subseteq S(i, (X_H, R, s_H))$. For the second relation, let $a \in S(i, (X_H, R, s_H))$ then either (i) a is eliminated by UF in $(X_H, R_{-i}, s_H - \mathbf{1})$ in which case, demand for a exceeds its supply (zero) at some application of the UF procedure to $(X_H, R_{-i}, s_H - \mathbf{1})$ which implies that there is $j \neq i$ and $a \; P_j \; 0$ or (ii) a is scarce for some $j \neq i$ in $(X_H, R_{-i}, s_H - \mathbf{1})$, in which case $a \; P_j \; 0$ because $S(j, (X_H, R_{-i}, s_H - \mathbf{1})) \subseteq U_0(R_j)$.

4.3 Optimality of Mechanism ψ^*

It is illustrative to compare mechanism ψ^* with other strategy-proof and envy-free mechanisms. Consider the following mechanism which can also be seen as a straightforward generalization of φ^* to the case with multiple supplies. For each (X_H, R, s_H) and each $i \in N$, $\psi_i(X_H, R, s_H) = f(R_i, Y(R, s_H))$ where $Y(R, s_H) \equiv \left\{a \in \widetilde{X}_H : |\{j \in N : a \; R_j \; 0\}| \leq s_{H,a}\right\}$. It is easy to check that ψ is strategy-proof and envy-free. Indeed, ψ is Pareto inferior to ψ^*. We contrast the two mechanisms' outcomes in the next example.

Example 1 *Let* $N = \{1, 2, 3\}$, $X_H = \{a, b, c, d, e, g, x, y, z\}$, $s_H = (2, 2, 2, 2, 2, 1, 3, 3, 3)$ *and the preference profile be as follows.*

$$\begin{array}{lcccccccc} R_1: & \boxed{a} & b & d & e & [x] & c & 0 & \\ R_2: & \boxed{b} & c & e & d & g & [y] & a & 0 \\ R_3: & \boxed{c} & a & d & e & g & [z] & b & 0 \end{array}$$

The allocations recommended by mechanisms ψ^* and ψ are shown in boxes and brackets respectively. Clearly, the allocation recommended by ψ^* Pareto dominates that by ψ. For each of a, b, c, d, e, and g, the number of agents that prefer this object to the null exceeds its supply, hence none of these objects is allocated by mechanism ψ. Mechanism ψ^* however allocates each agent her most favorite. This is because, none of a, b and c are scarce for 1, 2, and 3 in (X_H, R, s_H)

respectively. Observe that a is preferred to null by three agents, and its supply is only two. Based on this information, mechanism ψ behaves cautiously by not allocating a at all. On the other hand, for ψ^* this is not the case. This is because, only one of the agents, namely agent 2, prefers y to a, and y is abundant – in the sense that its supply is at least as many as the number of agents – in the problem (X_H, R, s_H). It is easy to check that it is this abundancy of y that prevents a being scarce for 1 in (X_H, R, s_H). The scarcity definition allows ψ^* to determine whether a possible conflict between no-envy and strategy-proofness arises by implicity working through the relevant subproblems of (X_H, R, s_H).

4.4 Which hypothetical set should be chosen?

One important question that remains to be answered is that given the large possibility of options the central planner faces to choose as a hypothetical set, which one(s) should she actually use?[14] While an exact answer to this question is beyond the scope of this paper,[15] we still offer a way to eliminate quite a few of these options. Figure 1 illustrates an example of a budget set that consists of all the possible hypothetical sets that a central planner can afford given his maximum spending. The figure shows three points on the frontier of the budget set, namely A, B, and C, and one interior point, A' each of which represents a different hypothetical set. The next lemma shows that a central planner, whose goal is to satisfy each unit's demands as much as possible given the budget constraint, can indeed do better than any interior point such as A' by instead choosing the corresponding outer point such as point A that is on the frontier of the budget set.

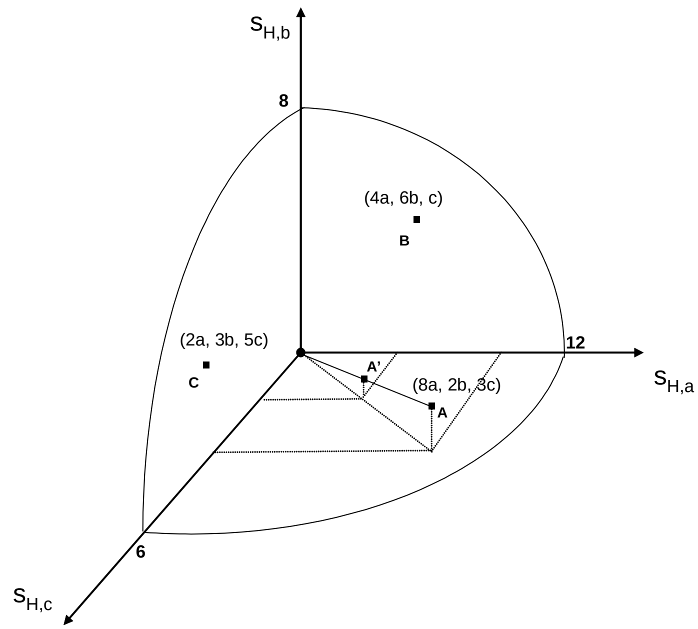
Remark 3 *Let (X_H, s_H) and (X_H, s'_H) be two hypothetical sets of objects such that $s'_H \geqq s_H$.*[16] *Then $\psi^*_i(X_H, R, s'_H)\ R_i\ \psi^*_i(X_H, R, s_H)$ for each $i \in N$ and each $R \in \mathcal{R}^N$.*

In other words, Remark 3 says that since mechanism ψ^* is monotonic with respect to the hypothetical set of objects, it suffices for the central planner to restrict attention to hypothetical sets that are on the outmost surface of the budget set (assuming away any complications due to the indivisibility of objects). Monotonicity of ψ^* with respect to the hypothetical set of objects follows directly from $S(i, (X_H, R, s'_H)) \subseteq S(i, (X_H, R, s_H))$ for each $i \in N$ and $R \in \mathcal{R}^N$.

[14] An alternative way to model and approach this problem could be looking for mechanisms that would recommend both a hypothetical set of resources and an allocation for that set. However, one major shortcoming of such an approach is that a mechanism now needs to choose from different allocations each of which is recommended for a different hypothetical set, which may be hard to justify. Our way of solving this problem can probably be argued to be comparably easier to implement in practice.

[15] This is a problem that needs to be answered from a *social choice* point of view, whose answer depends on the objectives of the social planner.

[16] Vector inequalities: $s'_H \geqq s_H$ means $s'_{H,x} \geq s_{H,x}$ for each $x \in X_H$; $s'_H \geq s_H$ means $s'_H \geqq s_H$ and $s'_H \neq s_H$; $s'_H > s_H$ means $s'_{H,x} > s_{H,x}$ for each $x \in X_H$.



An example of a hypothetical budget set.

5 Conclusion

In this paper we have taken a new mechanism design approach to model the resource purchasing and subsequent allocation decisions of the planner of a centrally administered organization. We have looked for equitable mechanisms, and also for equitable mechanisms that are immune to strategic maneuvers. The proposed mechanisms φ^* and ψ^* stand out as the most appealing mechanisms in their corresponding settings. Our way of modeling this problem call for an intrepretation of these mechanisms analoguous to the well-known Vickrey-Groves mechanisms which satisfy efficiency and strategy-proofness at the cost of budget-unbalancedness. These mechanisms can also be seen to implicitly assume that the central planner is capable of introducing or collecting additional side payments after the allocation decision is made. In our case, to the contrary, the central planner is assumed to introduce or collect objects (instead of money) from a hypothetical set of attainable objects before the allocation decision is made. Both ideas serve the purpose of achieving efficiency. Another commonality of our mechanisms with the Vickrey-Groves mechanisms is that both kinds of mechanisms, when making an assignment decision for a particular unit (agent) make explicit use of all other units (agents) but the particular one in consideration.

For other indivisible good allocation problems such as *house allocation with existing tenants* (Abdulkadiroğlu and Sönmez [4]) and *school choice problems*, it is customary to assume that for each object, there is an exogenously given *priority order* over units that needs to be respected while assigning the objects. Given a collection of priority orders, a mechanism is said to be *fair* if no unit ever envies some other unit for an assignment for which it has higher priority. There is a well-known fair mechanism for this context: The *student-optimal stable mechanism (SOSM).*[17] Mechanism ψ^* we introduced and SOSM bear striking resemblances. First, both try to serve the best interests of the units. That is, both mechanisms try to assign units their favorite choices as much as possible. When there is competition for a particular object among a certain group of units, SOSM resolves this conflict using the *exogenous* priority structure, whereas mechanism ψ^*does this in an *endogenous* way using the preferences of all units but the competing ones. The second and indirect one is in terms of the properties they satisfy. SOSM is strategy-proof and fair and Pareto dominates any other strategy-proof and fair mechanism whereas mechanism ψ^* is strategy-proof and envy-free and Pareto dominates any other strategy-proof and envy-free mechanism.

We believe and hope that the approach taken in this paper will bring a new breath and an alternative perspective to indivisible good allocation problems.

6 The Appendix

We first give an example to illustrate the scarcity definition.

Example 2 *Let* $N = \{1,2,3,4,5\}$ *,* $X_H = \{a,b,c,d,e\}$ *,* $s_H = (1,2,3,3,5)$ *and the preference profile be as follows.*

R_1	R_2	R_3	R_4	R_5
a	b	c	d	e
e	a	b	c	d
0	0	0	0	b
				0

We now determine if $a \in S(1,(X_H,R,s_H))$. Note that $U_a(R_1) \cup \{a\} = \{a\}$.

Step 1: We determine if a *is eliminated by* UF *in* $(X_H, R_{-1}, s_H - 1)$.

The supply vector is $s_H - \mathbf{1} = (0,1,2,2,4)$ and the preference profile is as follows:

R_2	R_3	R_4	R_5
b	c	d	e
a	b	c	d
0	0	0	b
			0

1st application of UF **procedure to** $(X_H, R_{-1}, s_H - \mathbf{1})$: Units 2, 3,4 and 5 demand objects b, c, d and e respectively. Demand for none of these objects exceeds its supply, therefore none is eliminated by the UF procedure.

We conclude that a *is* **not** *eliminated by* UF *in* $(X_H, R_{-1}, s_H - 1)$.

[17] Its outcome is calculated via the well-known *deferred acceptance algorithm*. For the two-sided matching context, it yields the most preferred stable allocation for each agent. See Roth and Sotomayor [21] for a comprehensive account. This rule is central to all the papers mentioned in footnote 15.

Step 2: We determine if *a satisfies (ii) of the definition of scarcity of* a *for unit* 1 *in* (X_H, R, s_H). We determine if $a \in S(2, (X_H, R_{-1}, s_H - 1))$. Note that $U_a(R_2) \cup \{a\} = \{a, b\}$. We consider each object in $\{a, b\}$ one at a time and check if it satisfies either (i) or (ii) of the definition of scarcity of a for unit 2 in $(X_H, R_{-1}, s_H - \mathbf{1})$.

Substep 1: We first consider object b.

Substep 1 (i) We determine if b *is eliminated by* UF *in* $(X_H, R_{-12}, s_H - 2 * 1)$.

The supply vector is $s_H - 2 * \mathbf{1} = (-1, 0, 1, 1, 3)$ and the preference profile is as follows:

R_3	R_4	R_5
c	d	e
b	c	d
0	0	b
		0

1st application of UF **procedure to** $(X_H, R_{-12}, s_H - 2 * 1)$: Units 3, 4 and 5 demand objects c, d and e respectively. Demand for none of these objects exceeds its supply therefore none is eliminated by the UF procedure. Note that even though a is not demanded, it is eliminated by UF in $(X_H, R_{-12}, s_H - 2 * 1)$ because its supply is negative.

Substep 1 (ii) We determine if b *satisfies (ii) of the definition of scarcity of* a *for unit* 2 *in* $(X_H, R_{-1}, s_H - 1)$. We determine if $b \in S(3, (X_H, R_{-12}, s_H - 2 * \mathbf{1}))$. Note that $U_b(R_3) \cup \{b\} = \{b, c\}$. We consider each object in $\{b, c\}$ one at a time and check if it satisfies either (i) or (ii) of the definition of scarcity of b for 3 in $(X_H, R_{-12}, s_H - 2 * \mathbf{1})$.

Subsubstep 1: We first consider object c.

Subsubstep 1 (i): We determine if c *is eliminated by* UF *in* $(X_H, R_{-123}, s_H - 3 * 1)$.

The supply vector is $s_H - 3 * \mathbf{1} = (-2, -1, 0, 0, 2)$ and the preference profile is as follows:

R_4	R_5
d	e
c	d
0	b
	0

1st application of UF **procedure to** $(X_H, R_{-123}, s_H - 3 * 1)$: Units 4 and 5 demand d and e respectively. Demand for d exceeds its supply. Object d is eliminated by the UF procedure in its first application. Note that a and b are eliminated by the UF procedure because each has a negative supply no matter they are demanded or not.

2nd application of UF **procedure to** $(R_{-124}, s_H - 3 * \mathbf{1})$: Units 4 and 5 demand c and e respectively. Demand for c exceeds its supply. Object c is eliminated by the UF procedure in its second application.

Subsubstep 2: We next consider object b.

Subsubstep 2 (i) We determine if b *is eliminated by* UF *in* $(X_H, R_{-123}, s_H - 3 * 1)$.

We refer the reader to subsubstep 1 (i). *Object* b *is eliminated by* UF *in* $(X_H, R_{-123}, s_H - 3 * 1)$.

Subsubstep 1 and 2 show that $b \in S(3, (X_H, R_{-12}, s_H - 2 * \mathbf{1}))$. Substep 1 ends: *Object* b *satisfies (ii) of the definition of scarcity of* a *for unit* 2 *in* $(X_H, R_{-1}, s_H - 1)$.

Substep 2: We next consider object a.

Substep 2 (i) We determine if *a is eliminated by UF in* $(X_H, R_{-12}, s_H - 2*1)$.

We refer the reader to substep 1(i). *Object a* is *eliminated by UF in* $(X_H, R_{-12}, s_H - 2*1)$.

Substep 2 ends: *Object a satisfies (i) of the definition of scarcity of a for unit 2 in* $(X_H, R_{-1}, s_H - 1)$. Substep 1 and 2 show that $a \in S(2, (X_H, R_{-1}, s_H - \mathbf{1}))$. Step 2 also ends: $a \in S(1, (X_H, R, s_H))$.

It can be easily shown that $S(1, (X_H, R, s_H)) = \{a\}$, $S(2, (X_H, R, s_H)) = \varnothing$, $S(3, (X_H, R, s_H)) = \varnothing$, $S(4, (X_H, R, s_H)) = \varnothing$ and $S(5, (X_H, R, s_H)) = \varnothing$. Thus, $\psi^*(X_H, R, s_H) = (e, b, c, d, e)$.

We now present the proofs of the results in the main text. For convenience, we prove Proposition 2 before Proposition 1.

Proof of Proposition 2: It is obvious that Z satisfies the three properties. Suppose φ^* is not *strategy-proof.* Then there are $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$, $i \in N$, and $R'_i \in \mathcal{R}$ such that $\varphi^*_i(X_H, (R'_i, R_{-i}))$ P_i $\varphi^*_i(X_H, R)$. Let $a \equiv \varphi^*_i(X_H, (R'_i, R_{-i}))$. Since $a \ P_i \ \varphi^*_i(X_H, R)$, this means there is $j \in N\backslash\{i\}$ such that $a \in U_0(R_j)$. But then, $\varphi^*_i(X_H, (R'_i, R_{-i})) \neq a$. To see that φ^* satisfies *equal treatment of equals,* let $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$ and suppose there are $i, j \in N$ such that $R_i = R_j$. Then $U_0(R_i) = U_0(R_j)$. This means $\varphi^*_i(X_H, R) = \varphi^*_j(X_H, R)$. Claim 1 will be useful in proving the other statements in Proposition 2.

Claim 1: *Let φ be a mechanism satisfying strategy-proofness and equal treatment of equals. Then for each $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$ and each $i \in N$, $\varphi_i(X_H, R) \cap (\cup_{j\neq i} U_0(R_j)) = \varnothing$.*

Proof: Suppose there are $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$ and $i \in N$ such that $\varphi_i(X_H, R) \cap (\cup_{j\neq i} U_0(R_j)) \neq \varnothing$. Let $a \equiv \varphi_i(X_H, R))$. Note that $a \neq 0$. Then, either $a \ P_i \ 0$ or $0 \ P_i \ a$. Let $j \in N\backslash\{i\}$ be such that $a \in U_0(R_j)$. Let $R' \in \mathcal{R}^N$ be such that $R'_k = R_k$ for each $k \in N\backslash\{i\}$ and $R'_i = R_j$. By *equal treatment of equals,* $\varphi_i(X_H, R') = 0$. If $\varphi_i(X_H, R) = a \ P_i \ 0$, unit i gains by reporting R_i instead of R'_i, contradicting *strategy-proofness.* If $0 \ P_i \ \varphi_i(X_H, R) = a$, unit i gains by reporting R'_i instead of R_i,contradicting *strategy-proofness.*

Now since, for each $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$ and each $i \in N$, $\varphi^*_i(X_H, R) = f(R_i, \widetilde{X}_H \backslash \cup_{j\neq i} U_0(R_j))$, by Claim 1, φ^* Pareto dominates any other mechanism satisfying the two properties. To see that any mechanism $\varphi \neq Z$ satisfying the two properties Pareto dominates Z, suppose by contradiction that there are $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$ and $i \in N$ such that $0 \ P_i \ \varphi_i(X_H, R)$. Then suppose unit i reports $R'_i = R_j$ for some $j \in N\backslash\{i\}$ instead of R_i. Then, by *equal treatment of equals,* $\varphi_i((X_H, (R'_i, R_{-i})) = 0$, contradicting *strategy-proofness.*

Since φ^* is *strategy-proof* and satisfies *equal treatment of equals,* by Lemma 1, it is *envy-free.* By Lemma 2, it is *efficient.*

Q.E.D.

Proof of Proposition 1: It is obvious that Z satisfies the three properties. Let $\varphi \neq Z$ be another mechanism satisfying them. Then there are $(X_H, R) \in (\mathcal{X}, \mathcal{R}^N)$, $i \in N$, $a \in \widetilde{X}_H$, such that $\varphi_i(X_H, R) = a$. We claim that for each $j \in N$, $\varphi_j(X_H, R) \neq 0$. Indeed, if there is $j \in N\backslash\{i\}$ such that $\varphi_j(X_H, R) = 0$, then simply letting $R'_j = R_i$, we have $\varphi_i(X_H, (R'_j, R_{-j}) = \varphi_j(X_H, (R'_j, R_{-j})) = \varphi_j(X_H, R) = 0$, contradicting *nonbossiness*. Next, let $j \in N\backslash\{i\}$ and $\varphi_j(X_H, R) \neq a$. Let $b \equiv \varphi_j(X_H, R)$. Since φ satisfies *strategy-proofness* and *equal treatment of equals,* by Claim 1 in the proof of Proposition 2, $0 \ R_i \ b$. Let R'_i be such that $a \ R'_i \ b \ R'_i \ 0 \ R'_i \ x$

for each $x \in X_H \backslash \{a, b\}$. By *strategy-proofness,* $\varphi_i(X_H, R) = \varphi_i(X_H, (R'_i, R_{-i}))$. But, since $b \; R'_i \; 0$ and φ satisfies *strategy-proofness* and *equal treatment of equals,* $\varphi_j(X_H, R) \neq \varphi_j(X_H, (R'_i, R_{-i}))$, contradicting *nonbossiness*.

Q.E.D.

Proof of Lemma 2: Suppose φ is *envy-free* but not *efficient*. Then there are a problem (X_H, R, s_H) and an allocation $\beta \in \mathcal{A}(Y_\alpha, s_\alpha)$ where $\alpha \equiv \varphi(X_H, R, s_H)$ such that $\beta_i \; R_i \; \alpha_i$ for each $i \in N$ and $\beta_j \; P_j \; \alpha_j$ for some $j \in N$. Since $\beta \in \mathcal{A}(Y_\alpha, s_\alpha)$, there is a unit $k \neq j$ such that $\alpha_k = \beta_j$. But then, since $\varphi_k(X_H, R, s_H) = \alpha_k \; P_j \; \alpha_j = \varphi_j(X_H, R, s_H)$, j envies k under (X_H, R, s_H).

Q.E.D.

Proof of Proposition 3: We first prove the first statement in the theorem. Suppose there exist a problem (X_H, R, s_H) and units $i, j \in N$ such that $UF_j(X_H, R, s_H) \; P_i \; UF_i(X_H, R, s_H)$. This means that at the final application of the procedure, object $UF_j(X_H, R, s_H)$ is available. Since unit i does not demand it, this means that $UF_i(X_H, R, s_H) \; P_i \; UF_j(X_H, R, s_H)$.

To prove the second statement, suppose there exist another envy-free mechanism φ, a problem (X_H, R, s_H), and a unit $i \in N$ such that $\varphi_i(X_H, R, s_H) \; P_i \; UF_i(X_H, R, s_H)$. Note that at the last application of the UF procedure to (X_H, R, s_H), each unit is assigned its favorite object among the available ones. This means that object $\varphi_i(X_H, R, s_H)$ is removed from the set of available objects at some application of the procedure. If this happens when the procedure is applied for the first time, then φ is not envy-free. If it happens when the UF procedure is applied for the second time, this means that for φ to be envy-free, φ assigns some unit an object that was removed from the set of available objects when the UF procedure is applied for the first time. But this is not possible by the former statement. Continuing in this fashion, we conclude that φ cannot assign any object that is removed from the set of available objects at some application of the UF procedure.

Q.E.D.

The following remark will be repeatedly used in the following lemmata. It says that if an object is eliminated by UF in a subproblem I of a problem, then it is also eliminated by UF in any subproblem $I'$of I.

Remark 4 *Let a problem* (X_H, R, s_H) *be given and let* $a \in \widetilde{X}_H$. *Let* $N' \subset N$. *If* a *is eliminated by* UF *in* $(X_H, R_{-N'}, s_H - |N'| * \mathbf{1})$ *then for each* $N'' \subseteq N \backslash N'$, a *is eliminated by* UF *in* $(X_H, R_{-(N' \cup N'')}, s_H - |N' \cup N''| * \mathbf{1})$.

Lemma 4 will be used in the proof of Lemma 3 and to show the envy-freeness of ψ^*. Lemma 4 says that if an object a is scarce for a unit i in a problem then a is also scarce for i in any of its subproblems that is obtained by removing a unit (other than i) and reducing the supply of each object by 1.

Lemma 4 Let a problem (X_H, R, s_H) be given and $a \in \widetilde{X}_H$. Let $a \in S(i, (X_H, R, s_H))$. Then, for each $k \neq i$, $a \in S(i,(X_H, R_{-k}, s_H - \mathbf{1}))$.

Proof : We will first show that Lemma 4 holds for two unit problems. Let $N \equiv \{i, k\}$, a problem (X_H, R, s_H) and $a \in \widetilde{X}_H$ be given. Let $a \in S(i, (X_H, R, s_H))$. We will show that $a \in S(i, (X_H, R_{-k}, s_H - \mathbf{1}))$. Since $a \in S(i, (X_H, R, s_H))$, each $x \in U_a(R_i) \cup \{a\}$ is

either (i) eliminated by UF in $(X_H, R_{-i}, s_H - \mathbf{1})$

or (ii) scarce for unit k in $(X_H, R_{-i}, s_H - \mathbf{1})$

Let $x \in U_a(R_i) \cup \{a\}$.

Case 1: (i) holds for x.

By Remark 4, x is eliminated by UF in $(X_H, R_{-ik}, s_H - 2 * \mathbf{1})$.

Case 2: (ii) holds for x.

Since $x \in S(k, (X_H, R_{-i}, s_H - \mathbf{1}))$ and there is no remaining unit in $(X_H, R_{-ik}, s_H - 2 * \mathbf{1})$ for whom x may be scarce in $(X_H, R_{-ik}, s_H - 2 * \mathbf{1})$, x is eliminated by UF in $(X_H, R_{-ik}, s_H - 2 * \mathbf{1})$.

Thus, in each case, x satisfies (i) of the definition of scarcity of x for unit i in $(X_H, R_{-k}, s_H - \mathbf{1})$. Since x is arbitrary, this is true for each object in $U_a(R_i) \cup \{a\}$. Hence, $x \in S(i, (X_H, R_{-k}, s_H - \mathbf{1}))$.

Assume by induction that Lemma 4 holds for problems with a unit set of cardinality 3, ..., $|N| - 1$. We will show that it also holds for problems with a unit set of cardinality $|N|$.

Let a problem (X_H, R, s_H), $a \in X_H$, and $i, k \in N$, $i \neq k$ be given. Let $a \in S(i, (X_H, R, s_H))$. Since $a \in S(i, (X_H, R, s_H))$, each $x \in U_a(R_i) \cup \{a\}$ is

either (i) eliminated by UF in $(X_H, R_{-i}, s_H - \mathbf{1})$

or (ii) scarce for some unit l in $(X_H, R_{-i}, s_H - \mathbf{1})$.

We will show that $a \in S(i, (X_H, R_{-k}, s_H - \mathbf{1}))$. For this purpose, we show that each $x \in U_a(R_i) \cup \{a\}$ is

either (i) eliminated by UF in $(X_H, R_{-ik}, s_H - 2 * \mathbf{1})$.

or (ii) scarce for some unit m in $(X_H, R_{-ik}, s_H - 2 * \mathbf{1})$.

Let $x \in U_a(R_i) \cup \{a\}$.

Case 1: Object x is eliminated by UF in $(X_H, R_{-i}, s_H - \mathbf{1})$.

By Remark 4, x is eliminated by UF in $(X_H, R_{-ik}, s_H - 2 * \mathbf{1})$.

Case 2: Object x is scarce for some unit l in $(X_H, R_{-i}, s_H - \mathbf{1})$.

Subcase 1: $k = l$.

By definition of scarcity of an object for a unit in a problem, x is

either (i) eliminated by UF in $(X_H, R_{-ik}, s_H - 2 * \mathbf{1})$

or (ii) scarce for some unit j in $(X_H, R_{-ik}, s_H - 2 * \mathbf{1})$

If (ii) holds, let $m \equiv j$.

Subcase 2: $k \neq l$.

By the induction hypothesis, $x \in S(l, (X_H, R_{-ik}, s_H - 2 * \mathbf{1}))$.

Let $m \equiv l$.

In each case, x is

either (i) eliminated by UF in $(X_H, R_{-ik}, s_H - 2 * \mathbf{1})$

or (ii) scarce for some unit m in $(X_H, R_{-ik}, s_H - 2 * \mathbf{1})$. Thus, x satisfies either (i) or (ii) of the definition of scarcity of a for unit i in $(X_H, R_{-k}, s_H - \mathbf{1})$. Since x is arbitrary, this is true for each object in $U_a(R_i) \cup \{a\}$. Hence, $a \in S(i, (X_H, R_{-k}, s_H - \mathbf{1}))$.

Q.E.D.

Proof of Lemma 3: For each problem (X_H, R, s_H), we need to show that

1. No unit is assigned more than one object in (X_H, R, s_H)
2. No object a is assigned to more than $s_{H,a}$ units in (X_H, R, s_H).

It is easy to see that ψ^* assigns each unit at most one object. We prove the second statement. Suppose, by contradiction there are a problem (X_H, R, s_H), $a \in \widetilde{X}_H$ and $K \subseteq N$ such that

$K \equiv \{i \in N : \psi_i^*(X_H, R, s_H) = a\}$ and $|K| \geq s_{H,a}+1$. Let $k \in K$. Hence, $\psi_k^*(X_H, R, s_H) = a$ and $a \notin S(k, (X_H, R, s_H))$. Let $T \equiv \{i \in N \setminus \{k\} : a \ P_i \ \varnothing\}$. Note that $K \subseteq T \cup \{k\}$. This, together with $|K| \geq s_{H,a} + 1$ imply that $|T| \geq s_{H,a}$.
Let $T_1, T_2 \subseteq T$.
Let $T_1 \equiv \{i \in T :$ there is $x \in U_a(R_i)$ such that $x \notin S(i, (X_H, R_{-k}, s_H - \mathbf{1}))\}$ and
$T_2 \equiv \{i \in T : U_a(R_i) \subseteq S(i, (X_H, R_{-k}, s_H - \mathbf{1}))\}$. Note that $T_1 \cup T_2 = T$.

Let $i \in T_1$. By definition of T_1, there is $x \in U_a(R_i)$ such that $x \notin S(i, (X_H, R_{-k}, s_H - \mathbf{1}))$. By Lemma 4, $x \notin S(i, (X_H, R, s_H))$. This, together with $x \in U_a(R_i)$ imply that $\psi_i^*(X_H, R, s_H) \neq a$. Since i is arbitrary, for each $i \in T_1$, $\psi_i^*(X_H, R, s_H) \neq a$. Then $K \subseteq T_2 \cup \{k\}$.
Let $t \equiv |T_2|$. Note that $t \geq s_{H,a}$. By definition of T_2,

$$\text{For each } i \in T_2, U_a(R_i) \subseteq S(i, (X_H, R_{-k}, s_H - \mathbf{1})). \tag{1}$$

Let $j_1 \in T_2$. By (1) and Lemma 4,

$$\text{For each } i \in T_2 \setminus \{j_1\}, U_a(R_i) \subseteq S(i, (X_H, R_{-kj_1}, s_H - 2 * \mathbf{1})). \tag{2}$$

⋮

Let $j_{t-1} \in T_2 \setminus \{j_1, j_2, ..., j_{t-2}\}$. By $(t-1)$ and Lemma 4,

$$U_a(R_{j_t}) \subseteq S(j_t, (X_H, R_{-kj_1j_2 \cdots j_{t-1}}, s_H - t * \mathbf{1})). \tag{t}$$

Since $s_{H,a} \leq t$, $s_{H,a}-t-1 \leq -1$. Thus, a is eliminated by UF in $(X_H, R_{-kj_1j_2 \cdots j_{t-1}j_t}, s_H-(t+1)*\mathbf{1})$. Note that a satisfies (i) of the definition of scarcity of a for j_t in $(X_H, R_{-kj_1j_2 \cdots j_{t-1}}, s_H - t * \mathbf{1})$. This together with (t) imply that

$$a \in S(j_t, (X_H, R_{-kj_1 \ldots j_{t-1}}, s_H - t * \mathbf{1}). \tag{1*}$$

(1^*) and $(t-1)$ imply that

$$a \in S(j_{t-1}, (X_H, R_{-kj_1 \ldots j_{t-2}}, s_H - (t-1) * \mathbf{1})). \tag{2*}$$

⋮

$(t-1^*)$ and (1) imply that

$$a \in S(j_1, (X_H, R_{-k}, s_H - \mathbf{1})). \tag{t*}$$

Note that (t^*) implies that a satisfies (ii) of the definition of scarcity of a for unit k in (X_H, R, s_H). Note that $a \notin S(k, (X_H, R, s_H))$. The last two statements, together with the definition of an object being scarce for a unit in a problem imply that there is $x \in U_a(R_k)$ that satisfies neither (i) nor (ii) of the definition of scarcity of a for k in (X_H, R, s_H) but then $x \notin S(k, (X_H, R, s_H))$.

This, together with $x \in U_a(R_k)$ imply that $\psi_k^*(X_H, R, s_H) \neq a$. A contradiction.

Q.E.D.

Proof of Theorem 1: We first prove that ψ^* satisfies *strategy-proofness*. Suppose not. There are $i \in N$, a problem (X_H, R, s_H) and $R_i' \in \mathcal{R}$ such that
$\psi_i^*(X_H, (R_i', R_{-i}), s_H) = a$ and $a\ P_i\ \psi_i^*(X_H, R, s_H)$. Then, $a \in S(i, (X_H, R, s_H))$. By definition of scarcity of a for i in (X_H, R, s_H), a is

either (i) eliminated by UF in $(X_H, R_{-i}, s_H - \mathbf{1})$

or (ii) scarce for some unit j in $(X_H, R_{-i}, s_H - \mathbf{1})$.

Since this is true regardless of unit i's preference, a also satisfies either (i) or (ii) of the definition of scarcity of a for unit i in $(X_H, (R_i', R_{-i}), s_H)$. Note that $\psi_i^*(X_H, (R_i', R_{-i}), s_H) = a$. Thus, $a \notin S(i, (X_H, (R_i', R_{-i}), s_H))$. The last three statements, together with the definition of an object being scarce for a unit in a problem imply that there is $x \in U_a(R_i')$ which is

neither (i) eliminated by UF in $(X_H, R_{-i}, s_H - \mathbf{1})$

nor (ii) scarce for any unit in $(X_H, R_{-i}, s_H - \mathbf{1})$ but then $x \notin S(i, (X_H, (R_i', R_{-i}), s_H))$.

The fact that in each problem ψ^* assigns each unit its favorite object among the ones that are not scarce for him together with $x \notin S(i, (X_H, (R_i', R_{-i}), s_H))$ and $x \in U_a(R_i')$ imply that $\psi_i^*(X_H, (R_i', R_{-i}), s_H) \neq a$. A contradiction.

We prove the *envy-freeness* of ψ^* through Lemma 4 and Lemma 5. Lemma 5 states that if an object a is scarce for a unit i in a problem then a satisfies (ii) of the definition of scarcity of a for units different from i in the same problem.

Lemma 5 Let $a \in S(\ i, (X_H, R, s_H))$. Then, for each $k \in N$, $k \neq i$, a satisfies (ii) of the definition of scarcity of a for unit k in (X_H, R, s_H)[18].

Proof : Let $i, k \in N$, $i \neq k$ be given. Let $a \in S(i, (X_H, R, s_H))$. By Lemma 4, $a \in S(i, (X_H, R_{-k}, s_H - \mathbf{1}))$. When checking (ii) of the definition of scarcity of a for k in (X_H, R, s_H) for object a, let j in the definition be i. Thus for object a, (ii) of the definition of scarcity of a for unit k in (X_H, R, s_H) is satisfied.

Q.E.D.

We now prove that ψ^* is *envy-free*. Suppose not. There are units $i, k \in N$, $i \neq k$ such that $\psi_k^*(X_H, R, s_H) = a$ and $a\ P_i\ \psi_i^*(X_H, R, s_H)$. Then, $a \in S(i, (X_H, R, s_H))$. By Lemma 5, a satisfies (ii) of the definition of scarcity of a for k in (X_H, R, s_H). Since $\psi_k^*(X_H, R, s_H) = a$, $a \notin S(k, (X_H, R, s_H))$. This, together with the fact that a satisfies (ii) of the definition of scarcity of a for unit k in (X_H, R, s_H), imply that there is $x \in U_a(R_k)$ such that both (i) and (ii) of the definition of scarcity of a for i in (X_H, R, s_H) fail. But this implies that $x \notin S(k, (X_H, R, s_H))$. This, together with $x \in U_a(R_k)$ imply that $\psi_k^*(X_H, R, s_H) \neq a$. A contradiction.

Q.E.D.

Lemma 6 is the key to prove that ψ^* is Pareto dominant in the class of *strategy-proof* and *envy- free* mechanisms. Lemma 6 states that no *envy-free* and *strategy-proof* mechanism ever assigns a unit an object that is scarce for it.

[18]Note that the lemma does not say that a is scarce for unit k in (X_H, R, s_H). For this to be true for each $x \in U_a(R_k) \cup \{a\}$, either (i) or (ii) of the definition of scarcity of a should be satisfied. The lemma says that the above statement holds for object a and doesn't say anything about objects in $U_a(R_k)$.

Lemma 6 No envy free and strategy-proof mechanism assigns a unit an object that is scarce for it.

Sketch of the proof: Before formally proving Lemma 6, we present a sketch of the proof for a particular problem and a particular envy-free and strategy-proof mechanism.

Step 1: Consider the problem in Example 2. Following the same steps in Example 2 we obtain that $a \in S(1,(X_H, R, s_H))$. Suppose, by contradiction that there exists an *envy-free* and *strategy-proof* mechanism φ such that $\varphi_1(X_H, R, s_H) = a$.

Let R_1' be as shown below. Let $x_0 = a$. By *strategy-proofness*, $z_0 \equiv \varphi_1(X_H,(R_1', R_{-1}), s_H) = a$. Hence, $s_a \geq 1$. Since $a \in S(1,(X_H, R, s_H))$, a is either eliminated by UF or scarce for some unit in $(X_H, R_{-1}, s_H - \mathbf{1})$. Hence, a satisfies either (i) or (ii) of the definition of scarcity of a in $(X_H,(R_1', R_{-1}), s_H)$. In Example 2 we determined that $a \in S(2,(X_H, R_{-1}, s_H - \mathbf{1}))$. Since a is the favorite object of 1 under R_1', $a \in S(1,(X_H,(R_1', R_{-1}), s_H))$. The preference profile is as follows:

R_1'	R_2	R_3	R_4	R_5
$\boxed{x_0}$	**b**	c	d	e
x_1	**a**	b	c	d
x_2	0	0	0	b
$\vdots$				0

We change the next unit's preference. We noted above that $z_0 = a \in S(2,(X_H, R_{-1}, s_H - \mathbf{1}))$. By *no-envy*, $z^1 \equiv \varphi_2(X_H,(R_1', R_{-1}), s_H) \in U_a(R_2) \cup \{a\} = \{a, b\}$. Suppose $z^1 = b$. (One can come up with the contradiction easily in the other case). Let $x_1 = z^1$. Let $N_1' \equiv N_0' \cup \{2\} = \{1, 2\}$ and $X_1 \equiv X_0 \cup \{x_1\} = \{a, b\}$. Note that $|N_1'| = 2$. Let $R_2' = R_1'$. By *strategy-proofness*, $z_1 \equiv \varphi_2(X_H,(R'_{N_1'}, R_{-N_1'}), s_H) \in U_b(R_2') \cup \{b\} = X_1 = \{a, b\}$.
By *no-envy*, $\varphi_l(X_H,(R'_{N_1'}, R_{-N_1'}), s_H) = z_1$ for each $l \in N_1'$. Hence, $s_{z_1} \geq 2$. Suppose $z_1 = b$. (Otherwise if we assume $z_1 = a$, the contradiction is immediate). The preference profile is as follows:

R_1'	R_2'	R_3	R_4	R_5
a	a	**c**	d	e
$\boxed{b}$	$\boxed{b}$	**b**	c	d
x_2	x_2	0	0	b
$\vdots$	$\vdots$			0

Since $z_0 = a \in S(2,(X_H, R_{-1}, s_H - \mathbf{1}))$, by Remark 1,
$(U_a(R_2) \cup \{a\}) = \{a, b\} \subseteq S(2,(X_H, R_{-1}, s_H - \mathbf{1}))$. Hence, each of objects a and b is either eliminated by UF or scarce for some unit in $(X_H, R_{-12}, s_H - 2*\mathbf{1})$. Thus $\{a, b\} \subseteq S(2,(X_H,(R_2', R_{-12}), s_H))$. Indeed in Example 2 we determined that a is eliminated by UF in $(X_H, R_{-12}, s_H - 2*\mathbf{1})$ and $b \in S(3,(X_H, R_{-12}, s_H - 2*\mathbf{1}))$.

We now continue to change the next unit's preference. By *no-envy*,
$z^2 \equiv \varphi_3(X_H,(R'_{N_1'}, R_{-N_1'}), s_H) \in U_b(R_3) \cup \{b\} = \{b, c\}$. Assume $z^2 = c$. (One can come up with the contradiction easily in the other case). Let $x_2 = z^2$. Let $N_2' \equiv N_1' \cup \{3\} = \{1, 2, 3\}$

and $X_2 \equiv X_1 \cup \{c\} = \{a, b, c\}$. Note that $|N'_2| = 3$. Let $R'_3 = R'_2$. By *strategy-proofness*, $z_2 \equiv \varphi_3(X_H, (R'_{N'_2}, R_{-N'_2}), s_H) \in U_c(R'_3) \cup \{c\} = X_2 = \{a, b, c\}$.
By *no-envy*, $\varphi_l(X_H, (R'_{N'_2}, R_{-N'_2}), s_H) = z_2$ for each $l \in N'_2$. Hence, $s_{z_2} \geq 3$. Suppose $z_2 = c$. (Contradiction is immediate for the other cases). The preference profile is as follows:

R'_1	R'_2	R'_3	R_4	R_5
a	a	a	**d**	e
b	b	b	**c**	d
$\boxed{c}$	$\boxed{c}$	$\boxed{c}$	0	b
x_3	x_3	x_3		0
$\vdots$	$\vdots$	$\vdots$		

Since $z_1 = b \in S(3, (X_H, R_{-12}, s_H - 2 * \mathbf{1}))$, by Remark 1,
$(U_b(R_3) \cup \{b\}) = \{b, c\} \subseteq S(3, (X_H, R_{-12}, s_H - 2 * \mathbf{1}))$. Hence, $z_2 = c$ is either eliminated by UF or scarce for some unit in $(X_H, R_{-123}, s_H - 3 * \mathbf{1})$. But then c satisfies either (i) or (ii) of the definition of scarcity of c for 3 in $(X_H, (R'_3, R_{-123},), s_H - 2 * \mathbf{1})$. Indeed in Example 2, we determined that c is eliminated by UF in $(X_H, R_{-123}, s_H - 3 * \mathbf{1})$. To conclude the argument we need to show that each object in X_2 is either eliminated by UF or scarce for some unit in $(X_H, R_{-123}, s_H - 3 * \mathbf{1})$. (Because $\varphi_3(X_H, (R'_{N'_2}, R_{-N'_2}), s_H)$ could be any object in X_2). Since $\{a, b\} \subseteq S(2, (X_H, (R'_2, R_{-12}), s_H - \mathbf{1}))$, by Lemma 4, $\{a, b\} \subseteq S(2, (X_H, (R'_2, R_{-123}), s_H - 2 * \mathbf{1}))$. By $R'_2 = R'_3$, $\{a, b\} \subseteq S(3, (X_H, (R'_3, R_{-123}), s_H - 2 * \mathbf{1}))$. Hence each of a and b is either eliminated by UF or scarce for some unit in $(X_H, R_{-123}, s_H - 3 * \mathbf{1}))$. Indeed in Example 2, we determined that a and b are eliminated by UF in $(X_H, R_{-123}, s_H - 3 * \mathbf{1})$.

Step 1 ends because each object in X_2 is eliminated by UF in $(X_H, R_{-123}, s_H - 3 * \mathbf{1})$.

Step 2: Let Z be the set of all objects that are eliminated by UF in $(X_H, R_{-123}, s_H - 3 * \mathbf{1})$. We have $Z = \{a, b, c, d\}$. Note also that $X_2 \subset Z$.

We next determine the set of units $J \subseteq N \backslash N'_2 = \{4, 5\}$ each of which prefers some object x in Z to null under R, and when UF is applied to $(X_H, R_{-123}, s_H - 3 * \mathbf{1})$ gets an object that is less preferred than x. We have $c \; P_4 \; 0$, $UF_4(X_H, R_{-123}, s_H - 3 * \mathbf{1}) = 0 \in L_c(R_4)$ thus $4 \in J$. Objects b and d are the only objects that are in Z and preferred by 5 to null; $b \; P_5 \; 0$, $d \; P_5 \; 0$, $UF_5(X_H, R_{-123}, s_H - 3 * \mathbf{1}) = e$, $e \notin L_b(R_5)$ and $e \notin L_d(R_5)$. Thus $5 \notin J$. Hence, $J = \{3\}$. We now complete the preferences of $1, 2$ and 3 by letting all objects in $Z \backslash X_2$ be more preferred than 0 and less preferred than those in X_2 and letting 0 be more preferred than $X_H \backslash Z$. The preference profile is as follows:

R'_1	R'_2	R'_3	R_4	R_5
a	a	a	**d**	e
b	b	b	**c**	d
$\boxed{c}$	$\boxed{c}$	$\boxed{c}$	0	b
d	d	d		0
0	0	0		

Since $z_2 = c$ is eliminated by UF in $(X_H, R_{-123}, s_H - 3 * \mathbf{1})$ and $s_c - 3 \geq 0$ there is at least one unit in J that prefers c to null under R and is assigned by UF in $(X_H, R_{-123}, s_H - 3 * \mathbf{1})$ something less preferred than c. Unit 4 is the only candidate. By *no-envy*, $z^3 \equiv \varphi_4(X_H, (R'_{N'_2}, R_{-N'_2}), s_H) \in U_c(R_4) \cup \{c\} = \{c, d\}$. Assume $z^3 = d$. (The argument applies for the other case). Let

$N_3' \equiv N_2' \cup \{4\} = \{1,2,3,4\}$. Note that $|N_3'| = 4$. Let $R_4' = R_3'$. Since 4 prefers c to null under R_4 and UF assigns 4 in $(X_H, R_{-123}, s_H - 3 * \mathbf{1})$ an object that is less preferred than c, each object in $U_c(R_4) \cup \{c\}$ is eliminated by UF in $(X_H, R_{-123}, s_H - 3 * \mathbf{1})$. Thus $z^3 \in Z$ and $z^3\ P_4'\ 0$. By *strategy-proofness*, $z_3 \equiv \varphi_4(X_H, (R'_{N_3'}, R_{-N_3'}), s_H)\ P_4'\ 0$. Thus $z_3 \in Z$. By *no envy*, $\varphi_l(X_H, (R'_{N_3'}, R_{-N_3'}), s_H) = z_3$ for each $l \in N_3'$. Thus $s_{z_3} \geq 4$. Since $z_3 \in Z$, it is eliminated by UF in $(X_H, R_{-123}, s_H - 3 * \mathbf{1})$. This, together with $s_{z_3} \geq 4$ imply that there are at least 2 units in J that prefer z_3 to null under R and are assigned by UF in $(X_H, R_{-123}, s_H - 3 * \mathbf{1})$ something less preferred than z_3. Hence $|J| \geq 2$. A contradiction.

Proof of Lemma 6: First, we prove the following claim.

Claim 2: *Let a problem (X_H, R, s_H) and $a \in X_H$ be given. Let $N' \subset N$. If a is eliminated by UF in $(X_H, R_{-N'}, s_H - |N'| * 1)$ and $s_{H,a} - |N'| \geq 0$ then there is a unit $i \in N \backslash N'$ such that $a \in S(i, (X_H, R_{-N'}, s_H - |N'| * 1))$.*

Proof: Let a problem (X_H, R, s_H) and $a \in X_H$ be given. Let $N' \subset N$. Let a be eliminated by UF in $(X_H, R_{-N'}, s_H - |N'| * \mathbf{1})$ and $s_{H,a} - |N'| \geq 0$. Then there are at least $s_{H,a} - |N'| + 1$ units in $N \backslash N'$ that prefer a to *null* and are assigned by UF in $(X_H, R_{-N'}, s_H - |N'| * \mathbf{1})$ an object that a is preferred to. Note that $s_{H,a} - |N'| + 1 \geq 1$. Thus, there is at least one such unit in $N \backslash N'$. Let $i \in N \backslash N'$ be such an agent, i.e, $a\ P_i\ 0$ and $UF_i(X_H, R_{-N'}, s_H - |N'| * \mathbf{1}) \in L_a(R_i)$. Since $UF_i(X_H, R_{-N'}, s_H - |N'| * \mathbf{1}) \in L_a(R_i)$, each $x \in U_a(R_i) \cup \{a\}$ is eliminated by UF in $(X_H, R_{-N'}, s_H - |N'| * \mathbf{1})$ and this, together with Remark 4, imply that each $x \in U_a(R_i) \cup \{a\}$ is eliminated by UF in $(X_H, R_{-(N' \cup \{i\})}, s_H - (|N'| + 1) * \mathbf{1})$. Thus, each $x \in U_a(R_i) \cup \{a\}$ satisfies (i) of the definition of scarcity of a for i in $(X_H, R_{-N'}, s_H - |N'| * \mathbf{1})$.

Q.E.D.

Suppose by contradiction there exists an *envy-free* and *strategy-proof* mechanism φ, a problem (X_H, R, s_H), an object $a \in X_H$ and a unit $i \in N$ such that $\varphi_i(X_H, R, s_H) = a$ and $a \in S(i, (X_H, R, s_H))$.

For each $k \in N$, let R_k' be such that $x_0\ R_k'\ x_1\ R_k'\ x_2\ R_k' \ldots . x_p\ R_k'\ z\ R_k'\ 0\ R_k'\ z'$ for each $z \in Z \backslash \bigcup_{s=0}^{p} x_s$ and each $z' \in X_H \backslash (\bigcup_{s=0}^{p} x_s \cup Z)$. Preferences R_k' will be constructed in steps 1&2 below, and therefore objects $(x_s)_{s=0}^{p}$ and the set Z will be determined throughout the proof. Let $x_0 = a$.

Step 1: Let $j_0 \equiv i$. By *strategy-proofness*, $z_0 \equiv \varphi_{j_0}(X_H, (R'_{j_0}, R_{-j_0}), s_H) = x_0$. Hence, $s_{H,z_0} \geq 1$. Let $N_0' \equiv \{j_0\}$ and $X_0 \equiv \{x_0\}$.
Since $z_0 (= x_0) \in S(j_0, (X_H, R, s_H))$, it is either eliminated by UF or scarce for some unit in $(X_H, R_{-j_0}, s_H - \mathbf{1})$. Hence $z_0 (= x_0)$ satisfies either (i) or (ii) of the definition of scarcity of $z_0 (= x_0)$ in $(X_H, (R'_{j_0}, R_{-j_0}), s_H)$. Since $z_0 (= x_0)$ is the favorite object of j_0 under R'_{j_0}, the previous statement implies statement (0).

$$z_0 (= x_0) \in S(j_0, (X_H, (R'_{j_0}, R_{-j_0}), s_H)). \qquad (0)$$

By (0), $z_0 (= x_0)$ is
either (i) eliminated by UF in $(X_H, R_{-j_0}, s_H - \mathbf{1})$.
or (ii) scarce for some unit j in $(X_H, R_{-j_0}, s_H - \mathbf{1})$.

Note that $s_{H,z_0} - 1 \geq 0$. If (i) holds then by Claim 2, there is a unit $j \in N \backslash N'_0$ such that $z_0 (= x_0) \in S(j, (X_H, R_{-j_0}, s_H - \mathbf{1}))$, i.e., (ii) also holds.

Let $j_1 \in N \backslash N'_0$ be such that $z_0 (= x_0) \in S(j_1, (X_H, R_{-j_0}, s_H - \mathbf{1}))$. By *no-envy*, $z^1 \equiv \varphi_{j_1}(X_H, (R'_{j_0}, R_{-j_0}), s_H) \in U_{z_0}(R_{j_1}) \cup \{z_0\}$. Let $x_1 = z^1$. Note that $x_1 \in U_{z_0}(R_{j_1}) \cup \{z_0\}$. Let $N'_1 \equiv N'_0 \cup \{j_1\}$. Note that $|N'_1| = 2$. Let $X_1 \equiv X_0 \cup \{x_1\}$. By *strategy-proofness*, $z_1 \equiv \varphi_{j_1}(X_H, (R'_{N'_1}, R_{-N'_1}), s_H) \in X_1$. By *no-envy*, $\varphi_l(X_H, (R'_{N'_1}, R_{-N'_1}), s_H) = z_1$ for each $l \in N'_1$. Hence $s_{H,z_1} \geq 2$. By $z_0 (= x_0) \in S(j_1, (X_H, R_{-j_0}, s_H - \mathbf{1}))$ and Remark 1, $(U_{z_0}(R_{j_1}) \cup \{z_0\}) \subseteq S(j_1, (X_H, R_{-j_0}, s_H - \mathbf{1}))$. Specifically, $x_1 \in S(j_1, (X_H, R_{-j_0}, s_H - \mathbf{1}))$. Hence, x_1 satisfies either (i) or (ii) of the definition of scarcity of x_1 for j_1 in $(X_H, (R'_{j_1}, R_{-N'_1},), s_H - \mathbf{1})$. By (0), Lemma 4 and $R'_{j_1} = R'_{j_0}$, $z_0 (= x_0) \in S(j_1, (X_H, (R'_{j_1}, R_{-N'_1}), s_H - \mathbf{1}))$[19]. (Indeed, by (0) and Lemma 4, $z_0 (= x_0) \in S(j_0, ((R'_{j_0}, R_{-N'_1}), s_H - \mathbf{1}))$. This, together with $R'_{j_1} = R'_{j_0}$ imply the conclusion.) This, together with the previous statement imply statement (1).

$$X_1 \subseteq S(j_1, (X_H, (R'_{j_1}, R_{-N'_1}), s_H - \mathbf{1})). \tag{1}$$

Note that $z_1 \in X_1$. By (1), z_1 is
either (i) eliminated by UF in $(X_H, R_{-N'_1}, s_H - 2 * \mathbf{1})$.
or (ii) scarce for some unit j in $(X_H, R_{-N'_1}, s_H - 2 * \mathbf{1})$.

Note that $s_{H,z_1} - 2 \geq 0$. If (i) holds then by Claim 2, there is a unit $j \in N \backslash N'_1$ such that $z_1 \in S(j, (X_H, R_{-N'_1}, s_H - 2 * \mathbf{1}))$, i.e., (ii) also holds.

We continue applying the same argument. In general, for each $v \in \mathbb{N} \backslash \{0\}$, let $j_v \in N \backslash N'_{v-1}$ be such that $z_{v-1} \in S(j_v, (X_H, R_{-N'_{v-1}}, s_H - v * \mathbf{1}))$. By *no-envy*, $z^v \equiv \varphi_{j_v}(X_H, (R'_{N'_{v-1}}, R_{-N'_{v-1}}), s_H) \in U_{z_{v-1}}(R_{j_v}) \cup \{z_{v-1}\}$. If $z^v \notin X_{v-1}$, let $x_v = z^v$, otherwise let $x_v = x_{v-1}$. Note that $x_v \in U_{z_{v-1}}(R_{j_v}) \cup \{z_{v-1}\}$. Let $N'_v \equiv N'_{v-1} \cup \{j_v\}$. Note that $|N'_v| = v + 1$. Let $X_v \equiv X_{v-1} \cup \{x_v\}$. By *strategy-proofness*, $z_v \equiv \varphi_{j_v}(X_H, (R'_{N'_v}, R_{-N'_v}), s_H) \in X_v$. By *no-envy*, $\varphi_l(X_H, (R'_{N'_v}, R_{-N'_v}), s_H) = z_v$ for each $l \in N'_v$. Hence $s_{H,z_v} \geq v + 1$. By $z_{v-1} \in S(j_v, (X_H, R_{-N'_{v-1}}, s_H - v * \mathbf{1}))$ and Remark 1, $(U_{z_{v-1}}(R_{j_v}) \cup \{z_{v-1}\}) \subseteq S(j_v, (X_H, R_{-N'_{v-1}}, s_H - v * \mathbf{1}))$. Specifically $x_v \in S(j_v, (X_H, R_{-N'_{v-1}}, s_H - v * \mathbf{1}))$. Hence x_v satisfies either (i) or (ii) of the definition of scarcity of x_v for j_v in $(X_H, (R'_{j_v}, R_{-N'_v}), s_H - v * \mathbf{1})$. By $(v-1)$, Lemma 4 and $R'_{j_v} = R'_{j_{v-1}}$, $X_{v-1} \subseteq S(j_v, (X_H, (R'_{j_v}, R_{-N'_v}), s_H - v * \mathbf{1}))$. This, together with the previous statement imply statement (v).

$$X_v \subseteq S(j_v, (X_H, (R'_{j_v}, R_{-N'_v}), s_H - v * \mathbf{1})). \tag{v}$$

Note that $z_v \in X_v$. By (v), z_v is
either (i) eliminated by UF in $(X_H, R_{-N'_v}, s_H - (v+1) * \mathbf{1})$.
or (ii) scarce for some unit j in $(X_H, R_{-N'_v}, s_H - (v+1) * \mathbf{1})$.

Note that $s_{H,z_v} - (v+1) \geq 0$. If (i) holds then by Claim 2, there is some unit $j \in N \backslash N'_v$ such that $z_v \in S(j, (X_H, R_{-N'_v}, s_H - (v+1) * \mathbf{1}))$, i.e., (ii) also holds.

[19] This conclusion could be derived from $z_0 (= x_0) \in S(j_1, (X_H, R_{-j_0}, s_H - \mathbf{1}))$ and z_0 being j_1's favorite object under R'_{j_1}. Nonetheless, the argument given in the proof is preferred to have symmetry between arguments as we continue changing units' preferences.

Let $v \in \mathbb{N}$. Since $X_v \subseteq S(j_v, (X_H, (R'_{j_v}, R_{-N'_v}), s_H - v * \mathbf{1}))$, by Remark 2, for each $x \in X_v$ there is a subproblem of $(X_H, (R'_{j_v}, R_{-N'_v}), s_H - v * \mathbf{1})$, say P', such that j_v is not in the unit set of P' and x is eliminated by UF in P'. In other words, for each $x \in X_v$, the test of scarcity of x for j_v in $(X_H, (R'_{j_v}, R_{-N'_v}), s_H - v * \mathbf{1})$, at some point terminates in part (i) of the scarcity definition.

By the finiteness of N, we will have $f \in \mathbb{N}$, $f \leq n$ such that for each $x \in X_f$, x is eliminated by UF in $(X_H, R_{-N'_f}, s_H - (f+1) * \mathbf{1})$. Since $z_f \in X_f$, z_f is eliminated by UF in $(X_H, R_{-N'_f}, s_H - (f+1) * \mathbf{1})$ where $z_f \equiv \varphi_{j_f}(X_H, (R'_{N'_f}, R_{-N'_f}), s_H)$. By *no-envy*, $\varphi_l(X_H, (R'_{N'_f}, R_{-N'_f}), s_H) = z_f$ for each $l \in N'_f$. Hence, $s_{H,z_f} - (f+1) \geq 0$. Let $p \equiv f$.

Step 2: Let Z be the set of all objects that are eliminated by UF in $(X_H, R_{-N'_f}, s_H - (f+1) * \mathbf{1})$. We know from step 1 that Z is nonempty. Indeed $X_f \subseteq Z$. Let J be the set of all units in $N \backslash N'_f$ that prefer some object $x \in Z$ to *null* under R and are assigned by UF in $(X_H, R_{-N'_f}, s_H - (f+1) * \mathbf{1})$ an object that x is preferred to. Formally,

$$J \equiv \left\{ l \in N \backslash N'_f : \begin{array}{c} x \ P_l \ 0 \text{ and } UF_l(X_H, R_{-N'_f}, s_H - (f+1) * \mathbf{1}) \\ \in L_x(R_l) \text{ for some } x \in Z \end{array} \right\}$$

Let $g \equiv |J|$. Since z_f is eliminated by UF in $(X_H, R_{-N'_f}, s_H - (f+1) * \mathbf{1})$ and $s_{H,z_f} - (f+1) \geq 0$ there is at least one unit in J that prefers z_f to *null* under R and is assigned by UF in $(X_H, R_{-N'_f}, s_H - (f+1) * \mathbf{1})$ an object that z_f is preferred to. (Hence, J is nonempty.)

Let $j_{f+1} \in J$ be such that $z_f \ P_{j_{f+1}} 0$ and $UF_{j_{f+1}}(X_H, R_{-N'_f}, s_H - (f+1) * \mathbf{1}) \in L_{z_f}(R_{j_{f+1}})$. By *no-envy*, $z^{f+1} \equiv \varphi_{j_{f+1}}(X_H, (R'_{N'_f}, R_{-N'_f}), s_H) \in U_{z_f}(R_{j_{f+1}}) \cup \{z_f\}$. Let $N'_{f+1} \equiv N'_f \cup \{j_{f+1}\}$. Note that $\left|N'_{f+1}\right| = f+2$. Since $z^{f+1} \in U_{z_f}(R_{j_{f+1}}) \cup \{z_f\}$ and $UF_{j_{f+1}}(X_H, R_{-N'_f}, s_H - (f+1) * \mathbf{1}) \in L_{z_f}(R_{j_{f+1}})$, z^{f+1} is eliminated by UF in $(X_H, R_{-N'_f}, s_H - (f+1) * \mathbf{1})$. Hence, $z^{f+1} \in Z$. By *strategy-proofness*, $z_{f+1} \equiv \varphi_{j_{f+1}}(X_H, (R'_{N'_{f+1}}, R_{-N'_{f+1}}), s_H) \ P'_{j_{f+1}} 0$. Thus $z_{f+1} \in Z$. By *no-envy*, $\varphi_l(X_H, (R'_{N'_{f+1}}, R_{-N'_{f+1}}), s_H) = z_{f+1}$ for each $l \in N'_{f+1}$. Thus $s_{H,z_{f+1}} \geq f+2$. Since $z_{f+1} \in Z$, it is eliminated by UF in $(X_H, R_{-N'_f}, s_H - (f+1) * \mathbf{1})$. This, together with $s_{H,z_{f+1}} \geq f+2$ imply that there are at least two units in J that prefer z_{f+1} to *null* under R and are assigned by UF in $(X_H, R_{-N'_f}, s_H - (f+1) * \mathbf{1})$ an object that z_{f+1} is preferred to. Hence there is at least one such unit in $J \backslash \{j_{f+1}\}$.

In general, let $v \in \{1, .., g\}$. Let $j_{f+v} \in J \backslash \{j_{f+1}, .., j_{f+v-1}\}$ be such that $z_{f+v-1} P_{j_{f+v}} 0$ and $UF_{j_{f+v}}(X_H, R_{-N'_f}, s_H - (f+1) * \mathbf{1}) \in L_{z_{f+v-1}}(R_{j_{f+v}})$. By *no-envy*, $z^{f+v} \equiv \varphi_{j_{f+v}}(X_H, (R'_{N'_{f+v-1}}, R_{-N'_{f+v-1}}), s_H) \in U_{z_f+v-1}(R_{j_{f+1}}) \cup \{z_{f+v-1}\}$. Let $N'_{f+v} \equiv N'_{f+v-1} \cup \{j_{f+v}\}$. Note that $\left|N'_{f+v}\right| = f+v+1$. Since $z^{f+v} \in U_{z_f+v-1}(R_{j_{f+v}}) \cup \{z_{f+v-1}\}$ and $UF_{j_{f+v}}(X_H, R_{-N'_f}, s_H - (f+1) * \mathbf{1}) \in L_{z_{f+v-1}}(R_{j_{f+v}})$, z^{f+v} is eliminated by UF in $(X_H, R_{-N'_f}, s_H - (f+1) * \mathbf{1})$. Hence, $z^{f+v} \in Z$. By *strategy-proofness*, $z_{f+v} \equiv \varphi_{j_{f+v}}(X_H, (R'_{N'_{f+v}}, R_{-N'_{f+v}}), s_H) \ P'_{j_{f+v}} 0$. Thus $z_{f+v} \in Z$. By *no-envy*, $\varphi_l(X_H, (R'_{N'_{f+v}}, R_{-N'_{f+v}}), s_H) = z_{f+v}$ for each $l \in N'_{f+v}$. Thus $s_{H,z_{f+v}} \geq f+v+1$. Since $z_{f+v} \in Z$, it is eliminated by UF in $(X_H, R_{-N'_f}, s_H - (f+1) * \mathbf{1})$. This, together with $s_{H,z_{f+v}} \geq f+v+1$ imply that there are at least $v+1$ units in J that prefer z_{f+v} to *null* under R and are assigned by UF in $(X_H, R_{-N'_f}, s_H - (f+1) * \mathbf{1})$ an object that z_{f+v} is preferred to. Hence there is at least one such unit in $J \backslash \{j_{f+1}, .., j_{f+v}\}$. Hence, $|J| \geq v+1$.

Applying the same argument repeatedly one can show that when $v = g$, we have $|J| \geq g + 1$. A contradiction.

Q.E.D.

We finally prove that ψ^* Pareto dominates any other *envy-free* and *strategy-proof* mechanism. Suppose by contradiction there exist another *envy-free* and *strategy proof* mechanism ψ, a problem (X_H, R, s_H), and a unit $i \in N$ such that $\psi_i(X_H, R, s_H) = z_0$ and $z_0 \ P_i \ \psi_i^*(X_H, R, s_H)$. Then, $z_0 \in S(i, (X_H, R, s_H))$. By Lemma 6, $\varphi_i(X_H, R, s_H) \neq z_0$. A contradiction.

Q.E.D.

7 References

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