Carnegie Mellon University **MSCF Program** **Math Prep**

Some Review Problems

1. Find all real solutions x.

   (a) $3e^{-3x} + e^{3x} = 5$

   (b) $\ln(2x+1) - \ln x = 4$

2. Evaluate the limit, or explain why the limit does not exist.

   (a) $\lim_{x\to 0^+} (1 + 2\sin(3x))^{\frac{1}{x}}$

   (b) $\lim_{x\to 0} \frac{\ln(1-x)+x}{x^2}$

   (c) $\lim_{x\to 1} \frac{x^2-1}{x^3+x}$

   (d) $\lim_{x\to\infty} x \ln\left(\frac{x+2}{x}\right)$

3. Use the definition of derivative to find $f'(x)$ for $f(x) = \frac{3}{x^2}$.

4. Find $f'(x)$. (You do not need to use the definition of derivative.)

   (a) $f(x) = x^3 e^{-x^2}$

   (b) $f(x) = \frac{\sqrt{x}}{1+x^2}$

   (c) $f(x) = (1 + x^8)^x$

5. A box with open top is to have a volume of 8 m^3. The base is to be square. The material for the base costs twice as much per m^2 as the material for the sides. Find the dimensions of the box that will minimize the total cost of materials. (Assume that there is no waste of materials in constructing the box.)

6. Assume that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and that $f(1) = 5$, $f'(1) = 3$, and $f''(1) = -2$. Define $F : [0, \infty) \to \mathbb{R}$ by $F(x) = f(\sqrt{x})$. Find $F''(1)$.

7. Assume that $f : \mathbb{R} \to \mathbb{R}$ is three times differentiable and that $f(1) = 3$, $f'(1) = -3$, $f''(1) = 4$ and that $|f'''(x)| \leq 7$ for all $x \in [.98, 1]$. Find an approximate value for $f(.98)$. What can you say about the error in this approximation?

8. Find $f^{(31)}(0)$ for the function f defined by $f(x) = x^3 e^{-x^2}$.

9. Find the Taylor series about 0 for the function f defined by

$$f(x) = \ln(\sqrt{1 - x^4}), \quad x \in (-1, 1).$$

10. Assume that $\int_0^2 f(x)\,dx = 3$, $\int_1^2 f(x)\,dx = 1$, $\int_1^5 f(x)\,dx = 2$, and $\int_3^5 f(x)\,dx = 4$.

(a) Find $\int_2^3 f(x)\,dx$.

(b) Find $\int_6^9 f\left(\frac{x}{3}\right)\,dx$.

11. Evaluate each of the following indefinite integrals.

(a) $\int x\ln(x^2+1)\,dx$

(b) $\int \frac{x^2}{x+1}\,dx$

(c) $\int e^{-2x}(e^{7x}+e^{-x})\,dx$

(d) $\int \frac{x}{(x^2+1)^2}\,dx$

12. Evaluate each of the following definite integrals.

(a) $\int_0^1 x^3 e^{x^2}\,dx$

(b) $\int_1^4 \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2\,dx$

(c) $\int_0^2 \frac{x^2}{1+x^3}\,dx$

13. Consider the function $F:[1,\infty)\to\mathbb{R}$ defined by

$$F(x)=\int_x^\infty \frac{2+\sin t}{t^2}\,dt,\quad x\geq 1.$$

Determine whether or not the improper integral $\int_1^\infty F(x)\,dx$ converges. Explain.

14. Determine all values of p for which the improper integral

$$\int_0^\infty \frac{1}{(1+x^4)^{\frac{p}{2}}}\,dx$$

converges. Carefully justify all of your conclusions with inequalities and/or theorems.

15. Determine whether or not each improper integral converges. (A quick explanation is sufficient.)

(a) $\int_0^1 \frac{1}{x^{\frac{1}{3}}(x^2+1)}\,dx$

(b) $\int_0^\infty x^{100}e^{-\sqrt{x}}\,dx$

(c) $\int_0^1 \frac{1}{\ln x}\,dx$

(d) $\int_0^\infty \frac{x^2}{1+x^3}\,dx$

(e) $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

16. Find $F'(x)$ for each of the following. (You may leave un-evaluated integrals in your answer.)

(a) $F(x) = \int_{x^3}^{\infty} e^{-t^2}\, dt$

(b) $F(x) = \int_{1}^{\ln x} \sqrt{t^4 + x^3}\, dt$

(c) $F(x) = \int_{\sqrt{x}}^{\sin x} \ln(1 + t^{10})\, dt$

17. Compute the indicated partial derivative.

    (a) $f_{xy}(x, y)$ for $f(x, y) = 3x^5y^2 - xe^{y^2}$

    (b) $\frac{\partial w}{\partial s}$ for $w = (s^2 - t^2)^3 + e^{-4st}$

    (c) $u_2(x_1, x_2, x_3)$ for $u(x_1, x_2, x_3) = x_1^2x_2^3x_3^4 + (x_1^2 + x_1^3)^{100}$ (Here u_2 means the partial derivative of $u(x_1, x_2, x_3)$ with respect to the second argument, x_2.)

18. Assume that $f : \mathbb{R}^3 \to \mathbb{R}$ has continuous first-order partial derivatives and that $\vec{\nabla} f(1, 0, 1) = (5, -1, 3)$. Define $\vec{u} : \mathbb{R} \to \mathbb{R}^3$ by

$$\vec{u}(t) = (e^{2t}, t^3 - 3\sin t, e^{-t}),$$

and $F : \mathbb{R} \to \mathbb{R}$ by $F(t) = f(\vec{u}(t))$. Find $F'(0)$.

19. Let $\alpha > 0$ be a given constant and n be a given positive integer. Show that the function u defined by

$$u(t, x) = e^{-\alpha^2 n^2 t} \sin(nx)$$

is a solution of the one-dimensional heat equation $u_t(t, x) = \alpha^2 u_{xx}(t, x)$.

20. Consider the function f defined by

$$f(t, x, y) = (x^2 + t^2) \ln(1 + x^2 + y^3).$$

Assume that x and y are differentiable functions of t such that when $t = 1$ we have

$$x = 1, \quad y = 2, \quad \frac{dx}{dt} = 3, \quad \frac{dy}{dt} = -1.$$

Find the value at $t = 1$ of the total time derivative $\frac{d}{dt} f(t, x(t), y(t))$.

21. Write an equivalent double integral with the order of integration reversed.

    (a) $\int_{-1}^{1} \int_{x^2}^{1} f(x, y)\, dy\, dx$

    (b) $\int_{0}^{6} \int_{0}^{\frac{1}{3}y} f(x, y)\, dx\, dy$

22. Evaluate each double integral.

    (a) $\int_{0}^{1} \int_{0}^{\sqrt{y}} x^3y^2\, dx\, dy$

    (b) $\int_{0}^{2} \int_{0}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2}\, dy\, dx$